Suggestions, PS 1

3. Reed and Simon I.5. To show $\lim_{n\to\infty} d(x_n, y_n)$ exists for any two Cauchy sequences, show first that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(x_m, y_n) - d(x_m, y_m). \qquad (1)$$

Show

$$|d(x_m, y_n) - d(x_m, y_m)| \leq d(y_n, y_m)$$

This shows the right side of (1) goes to 0 as $m, n \to \infty$. Use this method to show that $d(x_n, y_n) - d(x_m, y_m) \underset{m,n\to\infty}{\longrightarrow} 0$. Thus $\{d(x_n, y_n)\}_n$ form a Cauchy sequence (as a sequence of numbers), and so have a limit. Show that this limit is the same if the Cauchy sequence x_n is replaced by x'_n and y_n is replaced by y'_n, as long as $d(x_n, x'_n) \to 0$ and $d(y_n, y'_n) \to 0$. This will show that the limit depends only on the equivalence classes of $\{x_n\}$ and $\{y_n\}$. Thus the collection $\tilde{M}$ of equivalence classes of sequences is well-defined, and the distance between two elements of $\tilde{M}$ can now be defined, as these are just two Cauchy sequences (how is this done?). Follow by showing completeness of the space $\tilde{M}$. Show that we can identify an element x of M with the sequence consisting only of x's, so that we can identify M as a subset of the new space $\tilde{M}$. Why is M dense in $\tilde{M}$?

5. An example of completeness: You may assume the fact from Taylor series that if $S_N(x) = \sum_{n=0}^{N} x^n/n!$, then $S_N(x)$ converges to e^x uniformly on the interval $[0, 1]$. Note then that e^x is not a polynomial and hence not in P. You may assume the fact that a sequence of functions can only converge to one function using the distance measure defined by the norm.

6. Some metrics: For ρ_2 you may assume (or prove) that for any $f(x)$, $\int |f|dx = 0$ iff $f = 0$ a.e. To show the triangle inequality for ρ_3, let $\| f \| = \left(\int_0^1 |f|^2 dx\right)^{1/2}$. You need to show

$$\| f_1 - f_3 \| \leq \| f_1 - f_2 \| + \| f_2 - f_3 \| .$$

This is equivalent to

(1) $$\| g_1 + g_2 \| \leq \| g_1 \| + \| g_2 \|$$

for arbitrary g_i. Without loss we can assume that g_1 and g_2 are non-negative functions (why?). Now multiply out the integrand defining the left side of (1), and use Hölder's inequality:

(2) $$\int |f \cdot g| dx \leq \| f \| \; \| g \| .$$

To prove Hölder's inequality, you may first assume that f and g are both positive (by taking absolute values) and that $\| f \|$ and $\| g \|$ are both 1, since otherwise we could redefine f and g by multiplying them by constants so as to make this true and to still have equality (2) (why?). Now use the fact that $f(x)g(x) \leq 1/2(f^2(x) + g^2(x))$ (show this is true for any pair of numbers) to complete the argument.