Suggestions - PS 2

1. Closed and open sets: The proof for the real line can work in general. Recall $\mathcal{O}$ open if every $x \in \mathcal{O}$ has a ball around it in $\mathcal{O}$, while F is closed if it contains its limit points. Note that if $\mathcal{O}$ is open and x is a limit point of ~$\mathcal{O}$, then there is a sequence $x_n \underset{n\to\infty}{\longrightarrow} x$ with $x_n \in\ \sim \mathcal{O}$. But if $x \in \mathcal{O}$, there would be a ball $B(x,r) \subset \mathcal{O}$, which would contradict this - why? If $\sim \mathcal{O}$ is closed, then show that if $x \in \mathcal{O}$, there's no sequence $x_n \in\ \sim \mathcal{O}$ which converges to x. Thus there must be a ball $B(x,r)$ about x which is in $\mathcal{O}$. Why?

2. Continuity: Use the definition of continuity. If f is continuous, assume for a contradiction there exists an open $\mathcal{O}$ for which $f^{-1}(\mathcal{O})$ is not open. Then there is a sequence in $\sim f^{-1}(\mathcal{O})$ which converges to a point in $f^{-1}(\mathcal{O})$ (why?). Thus there is a sequence in $\sim \mathcal{O}$ which converges to a point in $\mathcal{O}$. What is the contradiction? Conversely if the inverse image of every open set is open then if $\{x_n\}$ is a convergent sequence, it follows that x_n is eventually contained (for n sufficiently large) in every open set containing its limit x. What follows about the image of the sequence?

3. Lim sup and lim inf. One approach is to prove

$$\overline{\lim}\, a_k = \lim_{n\to\infty} \left(\sup_{k \geq n} a_k \right),$$

and a similar statement for $\underline{\lim}$.

4. I.14: Show that the class of sets E for which $f^{-1}(E)$ is Borel measurable forms a σ-algebra. Why must it contain the Borels?

6. Corollary to Thm. I.12: Given $\{f_n\}$ converging in L^1, construct a subsequence as in the theorem proof and show it converges the same way. To show the function g_∞ obtained satisfies $g_\infty(x) < \infty$ a.e. show otherwise $\int g_\infty(x)dx = \infty$.

7. I.18: (a): An open set in $\mathbb{R}$ is a countable union of disjoint open intervals. Show first the characteristic function of an interval can be approximated by a sequence of continuous functions. Show the characteristic function $\chi_{\mathcal{O}}(x)$ of an open $\mathcal{O}$ is a (finite or infinite) sum of characteristic functions of open intervals, and so can be approximated analogously. Thus show $\chi_{\mathcal{O}}(x)$ can be approximated by a sequence of continuous functions.

(c) Show any continuous function can be approximated arbitrarily well in L^1 by simple functions.