Suggestions, PS 5

1. (III.1) Note to show the triangle inequality, you need to use the definition of L^∞ norm as a supremum modulo sets of measure 0. Recall the L^∞ norm $\| f \|_\infty$ is the smallest constant C such that $|f| \le C$ except on a set of measure 0. To show $\| f+g \| \le \| f \| + \| g \|$, use this directly. If |f| and |g| are less than C_1 and C_2 almost everywhere, does it follow $|f+g| \le C_1 + C_2$ a.e.? For completeness, show Cauchy sequences in L^∞ converge (uniformly except on null sets) to other functions in L^∞. For such a countable sequence f_n, is it possible to throw away a countable collection of such null sets? What's left over?

2. (III.2) For part (a), to consider ℓ_p and c_0, how about sequences of rational numbers which are non-zero for a finite number of terms? To show ℓ_∞ is not separable, you might consider just sequences just with values of 0 or 1. What if you can show that all of these sequences are more than a distance of 1 from each other? How many such sequences are there?

3. (III.3) If X is complete, you can use the definition of absolute summability and summability to show an absolutely summable sequence is summable. Conversely, if every absolutely summable sequence is summable, you might let $\{x_n\}$ be Cauchy, and consider a subsequence $\{x_{n_i}\}_i$ such that $\| x_{n_i} - x_{n_{i-1}} \| \le 2^{-i}$. What happens if we set $y_1 = x_{n_1}$, and $y_i = x_{n_i} - x_{n_{i-1}}$ for $i > 1$?

4. (Completeness of L^p) Show absolutely summable sequences are summable. If $\{f_n\}$ is a sequence with $\sum_{n=1}^{\infty} \|f_n\| < \infty$, show first that $\sum_{n=1}^{\infty} |f_n(x)| \in L^p$ by noting $\|\sum_{n=1}^{M} |f_n(x)|\|_p \le \|\sum_{n=1}^{M} |f_n(x)|\|_p \le \sum_{n=1}^{\infty} \|f_n\|_p$ and using monotone convergence. Now show that $\sum_{n=1}^{M} f_n(x)$ converges to $\sum_{n=1}^{\infty} f_n(x)$ in L^p by using the Minkowski inequality again.

5. (III.5) Why is it sufficient to prove $\kappa(\mathbb{R})$ is dense in C_∞ and the latter is closed?

6. (III.11): Consider first sequences which are non-zero for a finite number of terms, and define λ to be 0 on these. For a sequence x, let p(x) = lim sup x. How does Hahn-Banach apply now? Why does the resulting linear functional satisfy both the lower and the upper bound required? Note $\overline{\lim_{n\to\infty}} -x_n = -\underline{\lim_{n\to\infty}}\, x_n$.