MA 717
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PROBLEM SET 7
Due Thurs. March 15

Lectures 11, 12

This problem set is different from most in that the problems are generally shorter, and there are more of them. There are 10 required problems in this problem set. Whether or not you work on the starred problems, please make note of their statements/conclusions.

Starred problems are optional (not more difficult, but they will not be graded). If you have additional time some of them are worth doing.

1. Properties of Fourier Transform: If $F(k) = \mathcal{F}(f(x))$, what is:
(a) $\mathcal{F}(f(x+c))$ for c a constant? This result is known as the *shift theorem.*
(b) $\mathcal{F}(f'(x))$? Here you may assume $f(x) \underset{x \to \pm\infty}{\longrightarrow} 0$ if you wish.

2. More properties: If $F(\omega)$ is the Fourier transform of $f(x) \in L^1(\mathbb{R})$, show that $i\frac{dF}{d\omega}$ is the Fourier transform of $xf(x)$. You may assume $xf(x)$ is absolutely integrable, $\int |xf(x)|dx < \infty$, since we need to define its Fourier transform in any case.

You may use the following theorem:
Theorem: *If a function $f(x,\omega)$ is given, and it and its x-derivatives are integrable with respect to ω, i.e.,*

$$\int |f(x,\omega)|d\omega < \infty, \qquad \int \left|\frac{\partial}{\partial x} f(x,\omega)\right| d\omega < \infty$$

then $\int f(x,\omega)d\omega$ is differentiable with respect to x, with

$$\frac{d}{dx}\int f(x,\omega)d\omega = \int \frac{d}{dx} f(x,\omega)d\omega.$$

3. Orthogonal projection onto a subspace V: Recall that given a Hilbert space H and a closed subspace V, for $f \in H$ if we write

$$f = v + v^{\perp}$$

where $v \in V$ and $v^{\perp} \in V^{\perp}$, then the operator P defined by

$$Pf = P(v + v^{\perp}) = v$$

is the *orthogonal projection* onto V.

(a) Prove that P is bounded.
(b) What is the norm $\|P\|$ (assume that $P \neq 0$)?

(c) Show that P and $(I - P)$ are *idempotent*, i.e., that $P^2 = P$ and similarly for $I - P$. Here I denotes the identity.
(d) If $V \subset L^2(\mathbb{R})$ is the class of even functions and $f \in L^2$, find Pf. (Justify your answer.)

4*. Peaked distributions and their Fourier transforms:
(a) If $f(x)$ is a function with unit area, show that the scaled and stretched function $\frac{1}{\alpha}f(x/\alpha)$ also has unit area.
(b) If $F(\omega)$ is the Fourier transform of $f(x)$, show that $F(\alpha\omega)$ is the Fourier transform of $\frac{1}{\alpha}f(x/\alpha)$.
(c) Show that part (b) implies that broadly spread functions have sharply peaked Fourier transforms near $\omega = 0$, and vice-versa.

5*. The Haar father wavelet (scaling function): Let $\phi(x) = \begin{cases} 1 \text{ if } & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$ denote the basic Haar pixel function.
(a) Verify that $\{\phi(x-k)\}_{k=-\infty}^{\infty}$ are an orthonormal family of functions.
(b) For a complex-valued function $g(x) = \sum\limits_{k=-\infty}^{\infty} a_k\phi(x-k)$ in the span of this family (with $a_k \in \mathbb{C}$), verify that

$$\|g\|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

6. Haar wavelet expansions: Consider the function f(x) = $\begin{cases} 3x + 1 \text{ if } 0 \leq x < 1 \\ 0 \text{ otherwise} \end{cases}$. Find the Haar wavelet expansion of this function in the form

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{jk}\, \psi_{jk}(x).$$

7. Direct differences: Recall we have defined for a space V and subspaces W_1 and W_2, that $W_1 = V \ominus W_2$ if and only if V is the orthogonal direct sum of

$$V = W_1 \oplus_\perp W_2. \tag{1}$$

We wish to show here that this operation is unique. Show this by proving that, given V and a subspace $W_2 \subset V$, if equation (1) holds, then W_1 is uniquely determined by this equation.

8*. Orthonormal expansions: In a Hilbert space show that if S = $\{v_k\}$ is an orthonormal set (either finite or infinite), then if u = $\sum\limits_k c_k v_k$, it follows that $\|u\|^2 = \sum\limits_k |c_k|^2$.

9. Characterization of orthogonal direct sums: Let V be an inner product space and assume that W_1 and W_2 are subspaces which are orthogonal. Show that if every v $\in$ V can be expressed as a sum of vectors in W_1 and W_2, then V must be the orthogonal direct sum of W_1 and W_2.

10. Properties of the W_j: Recall the construction in class of the spaces V_j and W_j. Show that for $j \neq j'$, we have $W_j \perp W_{j'}$, i.e., every vector in W_j is perpendicular to every vector in W_j'.

11*. A basic inequality: Given a measurable function f on a measure space M, show that if $A \subset M$ is measurable,

$$\left(\int_A f d\mu\right)^2 \leq \mu(A) \int_A |f|^2 d\mu$$

12. Prove the following:

Lemma: The Fourier transform of an integrable function is continuous.

13*. Commutativity of convolutions: Defining $f * g = \int_{-\infty}^{\infty} f(x-y)g(y)dy$, verify through a change of variables that $f * g = g * f$.

14. Convolutions and the Fourier transform: If $\mathcal{F}(f(x)) = \widehat{f}(\omega)$ and $\mathcal{F}(g(x)) = \widehat{g}(\omega)$, verify that

$$\mathcal{F}((f * g)(x)) = \sqrt{2\pi}\,\widehat{f}(\omega)\widehat{g}(\omega)$$

15*. Reflection and the Fourier transform: Show that

$$\mathcal{F}(f(-x)) = \overline{\widehat{f}}(\omega),$$

if $\widehat{f}$ is the Fourier transform of $f(x)$.

16. Integral of the wavelet: Recall we assume throughout the development of wavelet theory that the integral of the scaling function $\int \phi(x)dx \neq 0$.

(a) Recall from class that $\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2)$. Show that this implies that (assuming $\int |\phi(x)|dx < \infty$) $m_0(0) = 1$.

(b) Show that this implies that $m_0(\pi) = 0$.

(c) Now use the construction of the wavelet $\psi(x)$ to evaluate $\widehat{\psi}(0)$, assuming that $\int |\psi(x)|dx < \infty$.

(d) Show that this yields $\int \psi(x)dx = 0$.