MA 717
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Problem Set 8
Due Thurs. March 22

Lecture 13

This is a shorter problem set because of the snowstorm on Tuesday of last week. It may be turned in on Thursday, or optionally on Friday by 5 pm, in the envelope outside my door.

Starred problems are optional

1. Fourier transforms and decay rates: Prove the following theorem:

Theorem: (a) If a function $\psi(x)$ has n derivatives which are integrable and which go to 0 at ∞, then the Fourier transform satisfies

$$|\widehat{\psi}(\omega)| \le K(1+|\omega|)^{-n}. \tag{1}$$

Conversely, if (11) holds, then $\psi(x)$ has at least $n-2$ derivatives.
(b) Similarly, if $\widehat{\psi}(\omega)$ and its first n derivatives are integrable and go to 0 at ∞, then

$$|\psi(x)| \le K(1+|x|)^{-n}. \tag{2}$$

while if (11) holds, then $\widehat{\psi}(\omega)$ has at least $n-2$ derivatives.

2*. The interchange of integration and summation: Given a scaling function $\phi(x) \in L^2$, we have shown in class that the following condition is equivalent to the basis $\{\phi(x-k)\}$ for V_0 being orthonormal, i.e. $(\phi(x-k),\phi(x-\ell)) = 0$:

$$\sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega-2n\pi)|^2 = \frac{1}{2\pi}.$$

First, we assumed that $\{\phi(x-k)\}_k$ were orthonormal, so that

$$(\phi(x-k),\phi(x-\ell)) = 0$$

and so

$$(\mathcal{F}(\phi(x-k)), \mathcal{F}(\phi(x-\ell))) = 0,$$

giving

$$\int_{-\infty}^{\infty} e^{i\omega(\ell-k)}|\phi(\omega)|^2\, d\omega = 0$$

Thus we concluded that if m $\neq 0$,

$$\left(\dots \int_{-4\pi}^{-2\pi} + \int_{-2\pi}^{0\pi} + \int_{0\pi}^{2\pi} + \right) e^{im\omega}|\hat{\phi}(\omega)|^2 \, d\omega = 0.$$

Thus we had

$$0 = \sum_{n=-\infty}^{\infty} \int_{n\cdot 2\pi}^{(n+1)\cdot 2\pi} e^{im\omega}|\hat{\phi}(\omega)|^2 \, d\omega = \sum_{n=-\infty}^{\infty} \int_{0}^{2\pi} e^{im\omega}|\hat{\phi}(\omega-2n\pi)|^2 \, d\omega$$

$$= \int_{0}^{2\pi} e^{im\omega} \sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega-2n\pi)|^2 \, d\omega$$

At this point in the argument, justify the interchange of summation and integration by showing that the integral of the absolute value of the above integrand, i.e. $\int_0^{2\pi} \sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega-2n\pi)|^2 d\omega$ is finite (it is known from advanced calculus that this condition is enough to justify the interchange). To show the last thing, you may assume also (another basic fact) that integrals and sums can be interchanged if the integrand is positive; so interchange the two and rewrite the integral as one over $(-\infty,\infty)$ which is known to be finite (since $\phi \in L^2$).

3*. Example of orthogonal subspace decomposition:
(a) Show that every function f in $L^2[-\pi,\pi]$ can be uniquely written in the form $f = f_1 + f_2$, where f_1 is even and f_2 is odd.
(b) Show that f_1 and f_2 are orthogonal to each other.
(c) Thus conclude that $L^2[-\pi,\pi]$ is an orthogonal direct sum of odd and even functions.
(d) Show that if W is the even functions in L^2, then $W^{\perp}$ is exactly the set of odd functions.

4*. Integral of the scaling function: We show here that if ϕ is a scaling function for a multiresolution analysis (satisfying conditions (a) - (f) from class), then

$$\left| \int_{-\infty}^{\infty} \phi(x) \, dx \right| = 1. \tag{12}$$

Prove this as follows.
(a) Recall that

$$\sum_k |\widehat{\phi}(2\pi k + \omega)|^2 = \frac{1}{2\pi}.$$

Use this to show (with the results of a previous problem) that $|\widehat{\phi}(0)| = \frac{1}{\sqrt{2\pi}}$.
(b) Conclude that (12) holds.
(c) Since the properties of the scaling function and multiresolution analysis are not changed if we replace the scaling function $\phi(x)$ with $\beta\phi(x)$ for some complex number β with $|\beta| = 1$, show that we may without loss of generality assume that $\int_{-\infty}^{\infty} \phi(x) dx = 1$.

5*. Condition for orthonormal basis: Prove that for the Hilbert space $\mathcal{H}$, given a set $\{\psi_n\}_{n=0}^{\infty}$ of functions with norm 1, the following is true: A necessary and sufficient condition for $\{\psi_n\}$ to be an orthonormal basis is that for any $f \in \mathcal{H}$,

(1) $$\sum_{n=0}^{\infty} |\langle \psi_n, f\rangle|^2 = \|f\|^2.$$

6. Background vs. details: To understand better the role of MRA's in decomposing functions, consider the following example using Haar wavelets. Consider again the function

$$f(x) = \begin{cases} 3x + 1 \text{ if } 0 \le x < 1 \\ 0 \text{ otherwise} \end{cases}$$

whose Haar wavelet expansion we found earlier.

(a) Recall that, for example, we have

$$L^2(\mathbb{R}) = V_1 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \dots$$

Conclude we can find an expansion of f in the form

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \phi_{1k}(x) + \sum_{j=1,k=-\infty}^{\infty} a_{jk} \psi_{jk}(x).$$

(b) Find the coefficients a_k, and sketch the "background function"

$$v_1 = \sum_{k=-\infty}^{\infty} a_k \phi_{1k}(x).$$

(c) Similarly find and sketch the first detail function

$$w_1 = \sum_{k=-\infty}^{\infty} a_{1k} \psi_{1k}(x)$$

(d) Construct and sketch the second detail function

$$w_2 = \sum_{k=-\infty}^{\infty} a_{2k} \psi_{2k}(x)$$

(e) To see how background and details fit together in this MRA, compare the graphs of the background function v_1, with the functions $v_2 = v_1 + w_1$ and $v_3 = v_1 + w_1 + w_2$, which contain two levels of detail enhancement.

(f) Show from above that

$$f(x) = v_1 + w_1 + w_2 + \dots$$

These observations justify appropriateness of the term "MRA". Fancier versions of such multiresolution algorithms (along with compression routines) have made it possible to compress two hour high definition motion pictures into 15 gigabyte downloads.