Suggestions for PS 8

1. Fourier transform and decay rates: You could use integration by parts, showing

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\int \psi(x)e^{-i\omega x}dx = \frac{(-1)^n}{\sqrt{2\pi}}\int \psi^{(n)}(x)\frac{e^{-i\omega x}}{(-i\omega)^n}dx \qquad (*)$$

(be careful!). Show (1) is equivalent to

$$|\widehat{\psi}(\omega)| \le K|\omega|^{-n}$$

and $|\widehat{\psi}(\omega)|$ bounded. Why? You can thus define

$$g(\omega) = \begin{cases} |\omega|^{-n} \text{ if } |\omega| \ge 1 \\ 1 \text{ if } |\omega| < 1 \end{cases} .,$$

and show

$$g(\omega) \le C_1(1+|\omega|)^{-n};$$

Why is $|\widehat{\psi}(\omega)| \le C_2 g(\omega)$? Show this for $|\omega| < 1$ by showing $|\widehat{\psi}(\omega)| \le C_2$- what property of ψ does this follow from?.

Conversely if (1) holds show

$$\psi^{(n-2)}(x) = \frac{1}{\sqrt{2\pi}}\int \widehat{\psi}(\omega)(i\omega)^{n-2}e^{i\omega x}d\omega.$$

You can start by showing ψ has at least one derivative, using the definition of the derivative and dominated convergence. Justify interchangeability of any integrals and derivatives!

Since the integral converges and is well-defined (why?), show ψ has (at least) $n-2$ derivatives.

3. Example of orthogonal subspace decomposition:
(a) Let $f_1 = \frac{1}{2}(f(x) + f(-x))$. What should f_2 be?
(b) You can use an appropriate change of variables in the integral checking orthogonality.
(d) Show first if f is odd then $f \in W^\perp$; this follows from the fact that the product of an odd and an even function is odd, and integrates to 0. Then show that if $f \in W^\perp$ then f must be odd. To do this write $f = f_1 + f_2$ where f_1 is even and f_2 is odd. For any $g \in W$, we have $0 = (f,g) = (f_1 + f_2, g)$. Set $g = f_1$ to show f_1 must be 0. You can use the fact that if the integral of a non-negative function is 0, then the it is the 0 function.

5. Condition for orthonormal basis:
First show they must be orthonormal by showing that if two of the functions (say ψ_1 and ψ_2) are not orthogonal, then (1) is false with $f = \psi_1 + \psi_2$. Then show they are complete using the Pythagorean theorem (II.1).