MA 717
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Problem Set 9
Due Thurs. March 29

Lectures 14, 15

The midterm will be a 4-hour timed take-home test, and will be handed out on Thursday March 29. You should complete it in 4 hours and then seal it, returning it in class on the following Tuesday. It will be open book, notes, problem sets and solutions, but no other resources (people or internet) should be available. Please photocopy your PS 9 solutions so that you have them it to work with for the midterm.

1. Meyer wavelets: Prove that the Fourier transform $\widehat{\psi}(\omega)$ of the Meyer wavelet is infinitely differentiable.

2. The m_0 function: One choice of m_0 is obtained by writing out its (finite) Fourier series directly:

(2) $\quad m_0(\omega) = \frac{1}{8}\,[(1+\sqrt{3}) + (3+\sqrt{3})e^{-i\omega} + (3-\sqrt{3})e^{-2i\omega} + (1-\sqrt{3})e^{-3i\omega}]$

(As shown in class, we could use (1) to construct a scaling function $\hat{\phi}(\omega)$, and from that construct the Daubechies wavelet using the method discussed in class; you do not have to do this here.) Verify by direct calculation that m_0 satisfies the condition

$$|m_0^2(\omega)| + |m_0^2(\omega+\pi)| = 1.$$

You may use a computer algebra system if you like - this can also be checked without one.

3. Two dimensional wavelets:

(a) If $\{\phi_k\}_{k=-\infty}^{\infty} \cup \{\psi_{jk}\}_{j=0,\,k=-\infty}^{\infty}$ forms an orthonormal wavelet basis for $L^2(\mathbb{R})$, show that the following set of functions is orthonormal in

$$L^2(\mathbb{R}^2) = \{f(x,y) : \int\int |f(x,y)|^2 dxdy < \infty\} :$$

$$\{\phi_k(x)\phi_{k'}(y) : k, k' \in \mathbb{Z}\} \cup \{\phi_k(x)\psi_{k'j}(y) : k, k', j \in \mathbb{Z}\}$$

$$\cup \{\psi_{kj}(x)\phi_{k'}(y) : k, k', j \in \mathbb{Z}\} \cup \{\psi_{kj}(x)\psi_{k'j'}(y) : k, k', j, j' \in \mathbb{Z}\}.$$

(b) In the case of Haar wavelets, sketch the wavelets $\phi_0(x)\phi_0(y)$, $\phi_0(x)\psi_{00}(y)$, and $\psi_{00}(x)\psi_{00}(y)$.

4. Daubechies filters: For the Daubechies wavelet of order 2, recall we had

$$m_0(\omega) = \frac{1}{8}\,[(1+\sqrt{3}) + (3+\sqrt{3})e^{-i\omega} + (3-\sqrt{3})e^{-2i\omega} + (1-\sqrt{3})e^{-3i\omega}]$$

which yields the coefficients h_k of the filter in the form

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega}.$$

(a) One way of finding the values of the Daubechies wavelet $\phi(x)$ at integer values is by solving an eigenvalue problem, resulting from the identity

$$\phi(x) = \sum_k h_k \sqrt{2}\phi(2x-k). \tag{1}$$

Assuming we know that the Daubechies wavelet is non-zero only in $[0,3]$, form the vector $\phi_0 = (\phi(0), \phi(1), \phi(2), \phi(3))$, and reformulate (1) to obtain a matrix eigenvalue equation of the form $M\phi_0 = \phi_0$ with eigenvector ϕ_0 and eigenvalue 1.
(b) Solve the eigenvalue equation and find ϕ_0, either by hand or using the computer. (Hint: you can normalize your integer values $\{\phi(k)\}_k$ with the requirement $\sum\limits_k \phi(k) = 1$. This follows from the *Poisson summation formula,* which says that for any continuous square integrable function $\phi(x)$, $\sum\limits_{k=-\infty}^{\infty} \phi(k) = \sqrt{2\pi} \sum\limits_{\ell=-\infty}^{\infty} \widehat{\phi}(2\pi\ell)$. Note that we have shown that $\widehat{\phi}(2\pi k) = 0$ for $k \neq 0$, while $\sum\limits_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}$ implies that $|\widehat{\phi}(0)| = \frac{1}{\sqrt{2\pi}}$. Why? Why are we then free to choose $\widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}}$? If you wish you can cut down the size of your matrix for this part, by showing first that in fact $\phi(0) = \phi(3) = 0$, so the unknown vector can be reduced to the 2- vector $(\phi(1), \phi(2))$).
(c) Once we know ϕ_0, use (1) to find the values of ϕ at the dyadic points of order 1 by finding the vector

$$\phi_1 = (\phi(0), \phi(1/2), \phi(1), \phi(3/2), \ldots, \phi(3))$$

in terms of ϕ_0.
(d) Write down the matrix equation which relates ϕ_1 and ϕ_0. Find the coefficient matrix explicitly.
(e) Defining the dyadic values of ϕ generally in the vector

$$\phi_k = (\phi(0), \phi(1/2^k), \phi(2/2^k), \phi(3/2^k), \ldots, \phi(3)),$$

express ϕ_{k+1} in terms of ϕ_k.
(f) Express $\phi_k$in terms of ϕ_0 (you can write the relation as a matrix product).

5. Prove the last proposition in Section IV.1

6*. Prove the second proposition in section IV.3

7. Reed and Simon, problem IV.7.