Suggestions for PS 9:

1. Meyer wavelets: Recall the wavelet is

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{i\omega/2} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega| - 1\right)\right] \text{ if } 2\pi/3 \le |\omega| \le 4\pi/3 \\ e^{i\omega/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\omega| - 1\right)\right] \text{ if } 4\pi/3 \le |\omega| \le 8\pi/3 \\ 0 \quad \text{elsewhere} \end{cases}$$

with ν an infinitely differentiable non-negative function satisfying

$$\nu(x) = \begin{cases} 0 \quad \text{if } x \le 0 \\ 1 \text{ if } x \ge 1 \\ \text{smooth transition in } \nu \text{ from 0 to 1 as } x \text{ goes from 0 to 1} \end{cases}$$

and

$$\nu(x) + \nu(1 - x) = 1.$$

(a) First show all derivatives of ν vanish at $x = 0$ and $x = 1$. Use, for example, the fact that since all the derivatives are continuous,

$$\left.\frac{d^n}{dx^n}\nu(x)\right|_{x=0} = \left.\frac{d^n}{dx^n}\nu(x)\right|_{x=0^-} = 0.$$

(b) At points x other than where the definition of $\widehat{\psi}(x)$ changes, show $\widehat{\psi}$ is infinitely differentiable by noting that it's a composition of infinitely differentiable functions.
(c) At transition points such as $\omega = \frac{2\pi}{3}$, show

$$\left.\frac{d^n}{d\omega^n}\widehat{\psi}(\omega)\right|_{\omega=2\pi/3^-} = 0 = \left.\frac{d^n}{d\omega^n}\widehat{\psi}(\omega)\right|_{\omega=2\pi/3^+}$$

so that $\widehat{\psi}$ is also infinitely differentiable there.

6. Second proposition in IV.3: Bolzano-Weierstrass

7. IV.7: (a) Let $\mathcal{S} = \{[a,b) | \; a, b \in \mathbb{R}$. Why is $\mathcal{F}$ the arbitrary unions of sets in $\mathcal{S}$? Thus show $\mathcal{S}$ is a base for $\mathcal{F}$.

(b) For $x \in \mathbb{R}$, is $\mathcal{N} = \{ [x,x+1/n) \,|\, n = 1,2,3, \ldots \}$ a neighborhood base at x? Why does any open $\mathcal{O}$ containing x also contain an $S \in \mathcal{S}$ containing x?

(c) Try supposing a countable $\mathcal{B} = \{B_i\}$ forming a base with $a_i = \inf B_i$. Consider $[a, b)$, where $a \notin \{a_i\}$, and $b > a$. Why is this not a union of sets in $\mathcal{B}$?