Pointwise Wavelet Convergence in Besov and Uniformly Local Sobolev Spaces

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Abstract

We announce new conditions for uniform pointwise convergence rates of wavelet expansions in Besov and uniformly local Sobolev spaces.

1. Introduction

The wide applicability of wavelet approximation methods to diverse technological problems is due to their localization in time and frequency spaces, existence of fast algorithms and, in the case of the Daubechies wavelets, orthogonality and compact support. A natural question is, where and how fast do such wavelet expansions converge?

The first results on L_2 and L_∞ convergence rates of r-regular miltiresolution expansions were in [Ma, Me, W]. For an overview of convergence rates of wavelet expansions in Sobolev and Besov spaces, see the text [HKPT]. For recent results in non-linear approximation [De] and rates of convergence in threshholding algorithms for wavelet expansions, see [CDKP]. For a study of shift invariant spaces see [BDR]. Our approach imposes less restrictive assumptions on the scaling function and/or wavelet, and uses different techniques. Here we summarize previous results and announce new results for Besov and uniformly local Sobolev spaces.

In [KKR1, KKR2] the wavelet expansion analogue of the Carleson result that Fourier series converge almost everywhere on $\mathbb{R}^d$ is shown (it is easier to prove than the corresponding Fourier series result). The main ideas of the proof involve showing that the kernels associated with the projection operators of the multiresolution analyses (MRA) are bounded by an L_1 radially decreasing convolution kernel, something which fails to hold in Fourier theory.

The techniques of [KKR2] are used in the study of the rate of convergence of the error $\|f - P_n f\|_\infty$ associated with MRA projections P_n, for functions f in Sobolev

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spaces $H_2^s(\mathbb{R}^d)$ [KR1]. The MRA's are assumed to satisfy weak regularity assumptions, in the sense that the kernel of P_0 is bounded by a convolution kernel in the class $[RB]$ (which consists of functions with integrable radial decreasing majorants). This situation is more general than r-regularity and occurs when the scaling function $\phi(x)$ or the logarithmically weighted wavelets $\psi^\lambda(x)\ln(2+|x|)$ are in $[RB]$.

In [KR1] it is shown that convergence rates of wavelet and multiresolution expansions depend on smoothness of the expanded function f. Specifically, if s is not larger than a fixed parameter σ and $f \in H_2^s(\mathbb{R}^d)$ then the error of approximation is $O(2^{-n(s-d/2)})$, with d the dimension and n the number of scales used in the expansion. More precisely, the pointwise approximation condition can be stated as, for all $n \geq 0$,

$$\|f - P_n f\|_\infty \leq C\ 2^{-n(s-d/2)}\|f\|_{H_2^s} \tag{1.1}$$

whenever $d/2 < s < \sigma$. The convergence rate of (1.1) is expected [W].

In a sequel paper [KR2] we show that the index σ is actually sharp and depends on the MRA used in the expansion. The upper bound σ is related to the behavior of the Fourier transform $\widehat{\psi}$ of the wavelet near the origin, a condition which is essential for the proof of (1.1). We have shown that σ can be defined in four equivalent ways, involving the operator $I-P_0$, the Fourier transform of the scaling function ϕ or of the wavelet ψ, or the scaling function's symbol m_0. For function spaces with smoothness s greater than σ the rate of convergence "freezes" and fails to improve, no matter how large s is. Such behaviors are again expected for wavelet expansions. Again the key to these proofs is the L_1 bound on the reproducing kernel of the MRA.

The natural spaces for wavelets are Besov spaces, and here we will use embedding theorems [BL], [De] and our results for Sobolev spaces to derive optimal pointwise convergence rates for some classes of Besov spaces. Wavelet convergence questions in local spaces are also important because wavelets are local in nature, and we expect that global convergence properties which assume that the function f being expanded decays at infinity also hold locally, independently of the decay properties of f.

We also announce here optimal rates of convergence for other spaces of smooth functions, such as the uniformly local counterparts $H_{p,ul}^s(\mathbb{R}^d)$ of the standard Sobolev spaces H_p^s (definitions in Section 6), in the range $1 \leq p \leq \infty$. We show that the expected optimal rate of convergence $\sigma - d/2$ holds in these cases, but only under a decay assumption on the scaling function, namely $|\phi(x)| \leq C|x|^{-\sigma-d/2}$.

An overview of the organization of this paper is as follows. Basic definitions are given in Section 2. Section 3 contains known results on convergence rates of wavelet expansions for Sobolev and Besov spaces from [HKPT]. In Section 4 statements of the main pointwise convergence rate results for Sobolev spaces $H_2^s(\mathbb{R}^d)$ are given in order to provide a context for new results in Besov and uniformly local Sobolev spaces given in Sections 5 and 6. Section 7 gives examples involving convergence rates for Meyer, Battle-Lemariè and Daubechies wavelets.

2. Basic Definitions

For detailed definitions and theory of an MRA we refer to [Da]. An MRA is defined as an increasing sequence of subspaces $\{V_n\}$ of $L_2(\mathbb{R}^d)$ $(d \geq 1)$ such that

$$f(x) \in V_n \text{ iff } f(2x) \in V_{n+1},$$

the intersection of the spaces is $\{0\}$, the closure of their union is all of L_2, and V_0 is invariant under integer translations. It is also generally assumed (though we do not require it here) that there exists a function $\phi(x)$ (the *scaling function*) whose integer translates form an orthogonal basis (ONB) for V_0.

Let W_i be the orthogonal complement of V_i in V_{i+1}, i.e., $W_i = V_{i+1} \ominus V_i$, so that $V_{i+1} = V_i \oplus W_i$. From existence of ϕ it follows that there is a set of basic wavelets $\{\psi^\lambda(x)\}_{\lambda\in\Lambda}$ (with Λ a finite index set) such that $\psi^\lambda_{jk}(x) \equiv 2^{jd/2}\psi^\lambda(2^j x - k)$ $(j \in \mathbb{Z},\ k \in \mathbb{Z}^d)$ form an orthonormal basis for W_j for fixed j, and form an orthonormal basis for $L^2(\mathbb{R}^d)$ as λ, j, k vary. Our results will hold for any wavelet set $\{\psi^\lambda\}_\lambda$ related to W_0 whose translations and dilations form an orthonormal basis for $L_2(\mathbb{R}^d)$, regardless of how they are constructed (see [Da], Ch. 10; [Me]; [HW]).

It follows from the above definitions that there exist numbers $\{h_k\}_{k\in\mathbb{Z}^d}$ such that the scaling equation

$$\phi(x) = 2^d \sum_{k=-\infty}^{\infty} h_k \phi(2x - k) \tag{2.1}$$

holds. We define

$$m_0(\xi) \equiv \sum_{k=-\infty}^{\infty} h_k e^{-ik\xi} \tag{2.2}$$

to be the *symbol* of the MRA. Note m_0 satisfies $\widehat{\phi}(\xi) = m_0(\xi/2)\widehat{\phi}(\xi/2)$, where $\widehat{}$ denotes Fourier transform where our convention for the Fourier transform is

$$\widehat{\phi}(\xi) \equiv \mathcal{F}(\phi)(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} \phi(x) e^{-i\xi\cdot x} dx$$

where $\xi \cdot x$ is inner product.

Definitions 2.1: Define P_n to be the L_2 orthogonal projection onto V_n, with kernel (when it exists) $P_n(x, y)$. We define $P_0 = P$.

Given $f \in L_2$,
(i) the *multiresolution approximation* of f is the sequence $\{P_n f\}_n$;
(ii) the *wavelet expansion* of f is

$$\sum_{j;k;\lambda} a^\lambda_{jk} \psi^\lambda_{jk}(x) \sim f, \tag{2.3a}$$

with a^λ_{jk} the L_2 expansion coefficients of f, and $\sim$ denoting convergence in L_2;
(iii) the *scaling-wavelet expansion* of f is

$$\sum_k b_k \phi_k(x) + \sum_{j\geq 0;k;\lambda} a^\lambda_{jk} \psi^\lambda_{jk}(x) \sim f, \tag{2.3b}$$

where the b_k, a^λ_{jk} are L_2 expansion coefficients, and $\phi_k(x) = \phi(x-k)$.

We say such sums converge in any given sense (e.g., pointwise, in L_p, etc.) if the sums are calculated in such a way that at any stage in the summation there is a uniform bound on the range (largest minus smallest) of j values for which we have only a partial sum over k, λ.

Definitions 2.2: The *Sobolev space* $H^s_2(\mathbb{R}^d)$, $s \in \mathbb{R}$ is defined by

$$H^s_2(\mathbb{R}^d) \equiv \left\{ f \in L_2(\mathbb{R}^d) : \|f\|_{H^s_2} \equiv \sqrt{\int |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s d\xi} < \infty \right\}.$$

The *homogeneous Sobolev space* is:

$$\widetilde{H}^s_2(\mathbb{R}^d) \equiv \left\{ f \in L_2(\mathbb{R}^d) : \|f\|_{\widetilde{H}^s_2} \equiv \sqrt{\int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi} < \infty \right\}.$$

Note the spaces contain the same functions (by virtue of the fact that $\widetilde{H}^s_2$ is restricted to L^2). Only the norms differ, and the second space is incomplete as defined (its completion contains non-L_2 functions which grow at ∞).

We denote the space $\mathcal{F}H^s_2$ to be the Fourier transforms of functions in H^s_2, with the analogous definition for $\mathcal{F}\widetilde{H}^s_2$.

In general the *classical Sobolev space* H^s_p on $\mathbb{R}^d$ is defined by

$$H^s_p(\mathbb{R}^d) = \{ f \in L_p(\mathbb{R}^d) : (1-\Delta)^{s/2} f \in L_p(\mathbb{R}^d) \}$$

for $1 \le p \le \infty$, where Δ is the Laplacian, defined by

$$\mathcal{F}\Delta f(x) = -\sum_i \xi_i^2 (\mathcal{F}f)(\xi), \quad \text{and}$$

$\xi = (\xi_1, \ldots, \xi_d)$ denotes the Fourier variable dual to $x = (x_1, \ldots, x_d)$. The norm is $\|f\|_{H^s_p(\mathbb{R}^d)} = \|(1-\Delta)^{s/2} f\|_{L_p(\mathbb{R}^d)}$.

Finally, the *Sobolev space* $W^m_p(\mathbb{R}^d)$ is defined as

$$W^m_p(\mathbb{R}^d) = \left\{ f \in L_p(\mathbb{R}^d) : f^{(j)} \in L_p(\mathbb{R}^d),\ j = 0, 1, 2, \ldots, m \right\}$$

where $f^{(j)}$ is a weak j^{th} derivative. The norm for this space is

$$\|f\|_{W^m_p(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \|f^{(m)}\|_{L_p(\mathbb{R}^d)}.$$

When $s \in \{0, 1, 2, \ldots\}$ then $H^s_p = W^m_p$ if $1 < p < \infty$.

Definitions 2.3: A multiresolution analysis (MRA) or corresponding family of wavelets ψ^λ yields *pointwise order of approximation* (or *pointwise order of convergence*) $s > 0$ *in* H^r_2 if for any $f \in H^r_2$, the n^{th} order approximation $P_n f$ satisfies

$$\|P_n f - f\|_\infty = O(2^{-ns}), \tag{2.4}$$

as n tends to infinity, if $r - d/2 > 0$ (if $r - d/2 \leq 0$ the left side of (2.4) is in fact infinite for some f).

It yields *best pointwise order of approximation* (*or convergence*) $s > 0$ *in* H_2^r if s is the largest positive number such that (2.4) holds for all $f \in H_2^r$. If the supremum s of the numbers for which (2.4) holds is not attained, then we denote the best pointwise order of convergence by s^-. By convention best order of approximation 0 means that the supremum in (2.4) fails to go to 0; thus $s \geq 0$ by our definitions.

The MRA yields *optimal order of approximation* (*or convergence*) s if s is the best pointwise order of approximation for sufficiently smooth f, i.e. for $f \in H_2^r$ for sufficiently large r (i.e., for $r > R$ for some $R > 0$). Thus this order of convergence is the best possible order in any Sobolev space. We say $s = \infty$ if the best order of approximation in H_2^r becomes arbitrarily large for large r.

In addition, the notion of optimal order can be extended to any scale of spaces $\{X^s\}_{s>0}$ in the same way. In particular we will use this notion for the scale of uniformly local Sobolev spaces $H^s_{p,ul}$ in Section 6.

Definitions 2.4: A function $f(x)$ on $\mathbb{R}^d$ is *radial* if f depends on $|x|$ only. A real valued radial function is *radial decreasing* if $|f(x)| \leq |f(y)|$ whenever $|x| \geq |y|$.

A function $\phi(x)$ is in the radially bounded class $[RB]$ if $|\phi(x)| \leq \eta(|x|)$, with $\eta(\cdot)$ a decreasing function on $\mathbb{R}^+$, and $\eta(|x|)$ integrable in x.

We say $\phi(x) \in [RB(N)]$ if, $\phi(x) \in [RB]$ and in addition, we can choose a $\eta(x)$ such that

$$\int \eta(|x|)|x|^N dx < \infty.$$

With a slight abuse of terminology, a kernel $P(x, y)$ is in $[RB]$ if $|P(x, y)| \leq \eta(|x-y|)$, where $\eta(|x|)$ is, as above, decreasing in $|x|$ and integrable in x.

3. Basic Results for Sobolev and Besov Spaces [HKPT]

We cite a basic result on convergence of wavelet expansions in Sobolev spaces $W_p^m(\mathbb{R}) 1 \leq p \leq \infty$.

Theorem 3.1 [HKPT]: *For x on the real line $\mathbb{R}$, assume that the integer translates of a scaling function ϕ of an MRA generate an orthonormal system such that $\phi(x) \in [RB(\mathrm{N}+1)]$ for an integer $N \geq 0$. Assume that one or more of the following equivalent conditions hold:*
(a) $|m_0(\xi)|^2 = 1 + o(|\xi|^{2N})$ as $\xi \to 0$,
(b) $\int x^n \psi(x) dx = 0$ for $n = 0, 1, \ldots, N$, where ψ is the wavelet associated to ϕ,
(c) $\widehat{\phi}(\xi + 2k\pi) = o(|\xi|^N)$ as $\xi \to 0$ for all $k \neq 0$.
Then $f \in W_p^{N+1}(\mathbb{R})$ if and only if

$$\|P_j f - f\|_p = O(2^{-j(N+1)}) \text{ as } j \to \infty, \tag{3.1}$$

for any $p \in [1, \infty]$.

That is, for functions in $W_p^{N+1}(\mathbb{R})$ the rate of convergence for the multiresolution expansion in the p norm is $N+1$. In the next section we will present necessary and sufficient conditions for rates of sup-norm convergence ($p=\infty$) for functions in $H_2^s(\mathbb{R}^n)$ under the more minimal assumption that $\phi \in [RB]$. We remark, however, that extension of our results to $H_p^s,\ \ 1 \le p < \infty$ will be straightforward. We will also present new results for Besov and uniformly local spaces.

To define Besov spaces on $\mathbb{R}^n$, we proceed as follows ([BL]). (For alternative definitions of Besov spaces see [De], [HKPT].)

Definitions 3.2: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex with α_i non-negative integers, and define for a function f on $\mathbb{R}^n$

$$D^\alpha f(x) = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}} f(x).$$

We define the *Schwarz space of rapidly decreasing functions* on $\mathbb{R}^n$ by

$$\mathcal{S}(\mathbb{R}^n) = \{f(x) : \|(1+|x|)^b D^\alpha f\|_\infty < \infty \ \forall\ b > 0\}$$

where $\|\cdot\|_\infty$ denotes the essential supremum norm.

It can be shown that there exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ whose support supp ϕ satisfies

$$\text{supp } \phi = \{\xi | 2^{-1} \le |\xi| \le 2\},$$

and such that $\phi(x) > 0$ for $2^{-1} < |\xi| < 2$, and

$$\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1 \qquad (\xi \ne 0).$$

We then define functions $\phi_k(x)$ and $\psi(x)$ such that

$$\mathcal{F}\phi_k(\xi) = \phi(2^{-k}\xi) \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{3.1a}$$

$$\mathcal{F}\psi(\xi) = 1 - \sum_{k=1}^{\infty} \phi(2^{-k}\xi). \tag{3.1b}$$

Letting $\mathcal{S}'$ denote the dual space, we define for $f \in \mathcal{S}',\ s \in \mathbb{R},\ 1 \le p, q \le \infty$,

$$\|f\|_p^{s,q} \equiv \|\psi * f\|_p + \left(\sum_{k=1}^{\infty} (2^{sk}\|\phi_k * f\|_p)^q\right)^{1/q},$$

and the *Besov space* $B_p^{s,q}$ to be all functions for which this norm is finite.

The next theorem gives convergence rates of multiresolution approximations in Besov spaces.

Theorem 3.4 [HKPT]: *Let ϕ be a scaling function generating an MRA whose integer translates form an orthonormal system, $\phi \in [RB(\mathrm{N}+1)]$ for some integer $N \geq 0$. Assume that ϕ is $N+1$ times weakly differentiable, and that the derivative $\phi^{(N+1)}$ satisfies*

$$\text{ess sup}_x \sum_k |\phi^{(N+1)}(x-k)| < \infty.$$

Then for any $0 < s < N+1,\ 1 \leq p, q \leq \infty$, and any function $f \in L_p$, the following conditions are equivalent:
(a) $f \in B_p^{s,q}(\mathbb{R})$
(b) $\|P_j f - f\|_p = 2^{-js}\epsilon_j$ for $j = 0, 1, \ldots$, with $\{\epsilon_j\}_j \in l_q$
(c) $\|\beta\|_p < \infty$ and $\|\alpha_j\|_{l_p} = 2^{-j(s+1/2-1/p)}\epsilon'_j,\ j = 0, 1, \ldots$ where $\{\epsilon'_j\} \in l_q$, $\beta = \{b_k\}_{k=-\infty}^{\infty}$ and $\alpha_j = \{a_{jk}\}_{k=-\infty}^{\infty}$ (see 2.3(b)).

In Section 5 we will see that our conditions for convergence rates of expansions of functions in $H_2^s(\mathbb{R}^d)$, together with Besov-Sobolev embedding theorems [De], will lead to a supremum rate of convergence for functions in Besov spaces $B_2^{s,q}(\mathbb{R}^d)$ (Theorem 5.2 and Corollary 5.5).

4. Pointwise Convergence Theorems for Sobolev Spaces

The results below extend those presented in the previous section to necessary and sufficient conditions for given convergence rates, for expansions in Sobolev spaces on $\mathbb{R}^d$. We remark that these theorems deal with pointwise sup-norm (i.e. L_∞) convergence, but can be extended to convergence results of the same nature for L_p spaces.

The following theorem states that under mild assumptions on the MRA (i.e., that the scaling function or wavelet has a radially decreasing L_1 majorant), for $f \in H_2^s(\mathbb{R}^d)$ the rate of convergence to 0 of the error $\|f - P_n f\|_\infty$ has sharp order $2^{-n(s-d/2)}$. For the sake of brevity we have summarized four theorems into the following.

Theorem 4.1 [KR1]: *Given a multiresolution analysis with either*
(i) a scaling function $\phi \in [RB]$,
(ii) a family of basic wavelets $\psi^\lambda \in [RB]$ or
(iii) a kernel $P(x, y)$ for the basic projection P satisfying $|P(x, y)| \leq F(x - y)$ with $F \in [RB]$,
then the following conditions (a to e) are equivalent for $s > d/2$, with d the dimension:

(a) The multiresolution approximation yields pointwise order of approximation $s - d/2$ in H^s.

(b) The projection $I - P_n : \widetilde{H}_2^s \to L^\infty$ is bounded, with I the identity.

(c) For every family of basic wavelets corresponding to $\{P_n\}_n$, $\psi^\lambda \in \mathcal{F}\widetilde{H}^{-s}$ for each λ.

(c') For every such family of basic wavelets and for each λ,

$$\int_{|\xi|<\delta} |\widehat{\psi^\lambda}(\xi)|^2 |\xi|^{-2s} d\xi < \infty \tag{4.1a}$$

for some (or for all) $\delta > 0$.
(d) For every scaling function $\phi \in [RB]$ *corresponding to* $\{P_n\}_n$, $(1-(2\pi)^{d/2}|\widehat{\phi}|)^{1/2} \in \mathcal{F}\widetilde{H}_2^{-s}$.
(d′) For every such scaling function

$$\int_{|\xi|<\delta}(1-(2\pi)^{d/2}|\widehat{\phi}(\xi)|)|\xi|^{-2s}d\xi < \infty \tag{4.1b}$$

for some (or all) $\delta > 0$.
(e) For every symbol $m_0(\xi)$ *corresponding to* $\{P_n\}$

$$\int_{|\xi|<\delta}(1-|m_0(\xi)|^2)|\xi|^{-2s}d\xi < \infty \tag{4.1c}$$

for some (or all) $\delta > 0$.

Statements (c) and (c') are related to the vanishing moment properties of the wavelets ψ^λ around the origin, while (d) and (d') are related to so-called Strang-Fix conditions on the scaling function ϕ.

For the remainder of the paper we assume that one of the following conditions holds:

(i) The projection P onto V_0 satisfies $|P(x,y)| \leq F(x-y)$ for some $F \in [RB]$.
(ii) The scaling function $\phi \in [RB]$.
(iii) For a wavelet family ψ^λ, $\psi^\lambda(x)(\ln(2+|x|)) \in [RB]$ for all λ.

By representing the kernel of $P(x,y)$ in terms of sums of products involving ϕ or ψ^λ, it is shown in [KKR1] that *(ii)* $\Rightarrow$ *(i)* and *(iii)* $\Rightarrow$ *(i)*.

We refer to formulas (4.1a, b, c) as motivation for the following definitions.

Definitions 4.2: We define for $s, c \geq 0$

$$I_s(c) \equiv \int_{1\geq|\xi|\geq c}\left(1-(2\pi)^{d/2}|\widehat{\phi}(\xi)|\right)|\xi|^{-2s}d\xi \tag{4.2}$$

$$K_s(c) \equiv \sup_\lambda \int_{1\geq|\xi|\geq c}|\widehat{\psi^\lambda}(\xi)|^2|\xi|^{-2s}d\xi$$

$$M_s(c) \equiv \int_{1\geq|\xi|\geq c}(1-|m_0(\xi)|^2)|\xi|^{-2s}d\xi.$$

In this paper an often-used consequence of Theorem 4.1 is the existence of a least upper bound σ (best Sobolev parameter) on s, depending only on the MRA, for which *(b)* of Theorem 4.1 holds. This motivates the following definition.

Definition 4.3: The *best Sobolev parameter* σ of an MRA is

$$\sigma = \sup\{s > 0 | (I-P) : \widetilde{H}_2^s \to L_\infty \text{ is bounded}\}. \tag{4.3}$$

By convention $\sigma = 0$ if the set in the supremum is empty.

It can be shown that if the best Sobolev parameter $\sigma \neq 0$, then $\sigma > d/2$, and the set $\Sigma \equiv \{s > 0 | (I-P) : \widetilde{H}_2^s \to L^\infty \text{ is bounded}\}$ satisfies $\Sigma = (d/2, \sigma]$ or $\Sigma = (d/2, \sigma)$.

From Theorem 4.1 we have immediately:

Proposition 4.4: *If the best Sobolev parameter $\sigma \neq 0$, then*

$$\begin{aligned} \sigma &= \sup\{s > 0 | I_s(0) < \infty\} \\ &= \sup\{s > 0 | K_s(0) < \infty\} \\ &= \sup\{s > 0 | M_s(0) < \infty\}. \end{aligned} \tag{4.4}$$

In Theorem 4.1, σ is important in that all statements hold only if $s \leq \sigma$. For approximation rates in H_2^s, we have the following, which will appear in [KR2]. It summarizes convergence rates in all H_2^s in terms of properties of the projections P_n or of integrals involving the wavelets or scaling functions.

Theorem 4.5 [KR2]: *Given a multiresolution approximation $\{P_n\}$,*
(o) If $\sigma = 0$, there is no positive order of approximation for the MRA $\{P_n\}$ in any H_2^s, $s \in \mathbb{R}$.
If (o) does not hold then $\sigma > d/2$ and:
(i) For $0 \leq s \leq d/2$, the best pointwise order of approximation in H_2^s is 0;
(ii) If $d/2 < s < \sigma$, the best pointwise order of approximation in H_2^s is $r = s - d/2$;
(iii) If $s = \sigma$, the best pointwise order of approximation in H_2^s is

$$r = \begin{cases} \sigma - d/2 & \text{if } I_\sigma(0) < \infty \\ (\sigma - d/2)^- & \text{if } I_\sigma(0) = \infty \end{cases};$$

(iv) If $s > \sigma$, the best pointwise order of approximation in H_2^s is

$$r = \begin{cases} \sigma - d/2 & \text{if } I_{\sigma+1/2}(c) = O(1/c)\ (c \to 0) \\ (\sigma - d/2)^- & \text{otherwise} \end{cases};$$

(v) In (iii) and (iv) above, $I_s(c)$ can be replaced by $K_s(c)$ or by $M_s(c)$.

Another way to say *(iv)* is that if $s > \sigma$, then there exists $g \in H_2^s(\mathbb{R}^d)$ such that for all $\epsilon > 0$, $\sup_j 2^{j(\sigma+\epsilon-d/2)} \|g - P_j g\|_\infty = \infty$. This says the convergence rate cannot be improved for functions belonging even to very smooth Sobolev spaces, i.e., convergence rates are wavelet dependent. Moreover we note that the value $\sigma + 1/2$ used above in condition *(iv)* is not crucial for its statement. Equivalent conditions to those in *(iv)* exist in the form $I_{\sigma+\alpha/2}(c) = O(c^{-\alpha})$ for any (or all) $\alpha > 0$.

In terms of the Sobolev order s of the expanded function f and the best Sobolev parameter σ of the MRA, the diagram above gives rates for an MRA expansion in any Sobolev space (or local Sobolev space; see Section 6). The rates on the boundary region $s = \sigma$ in *(iii)* above are not indicated in the diagram.

We now state our result for optimal pointwise orders of convergence in Sobolev spaces. Recall σ denotes the best Sobolev parameter of $\{P_n\}$, and that optimal order

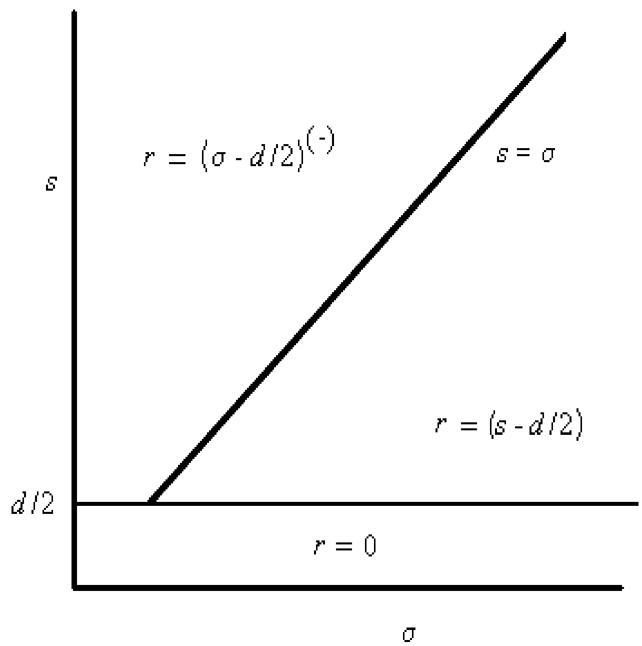

Figure 1: Approximation rate diagram; see Theorem 4.5 (*iii*) for rates on the boundary $s = \sigma$. The $(-)$ in $(\sigma - d/2)^{(-)}$ indicates that the superscript $-$ is present only in some cases.

of approximation denotes the highest order of approximation in sufficiently smooth Sobolev spaces.

Corollary 4.6: [KR2] *If the best Sobolev parameter* $\sigma \neq 0$*, then the wavelet collection* ψ^λ *[or scaling function* ϕ*] yields optimal pointwise order of approximation:*

(i) $\sigma - d/2$ *if* $K_{\sigma+1/2}(c) = O(1/c)$ *[where* K *can be replaced by* I *or* M*], and*

(ii) $(\sigma - d/2)^-$ *otherwise.*

This optimal order is attained for all functions f *with smoothness greater than* σ*, i.e., for* $f \in H_2^s$ *with* $s > \sigma$*.*

Corollary 4.6 gives "best possible" pointwise convergence rates, i.e., convergence rates for the smoothest possible functions. In fact this optimal rate is largely independent of how smoothness is defined, i.e., which particular scale of spaces we are working with. Such a statement is possible because when the smoothness parameter s is sufficiently large, the most used scales of "smoothness spaces" satisfy inclusion relations. For example for s' large the space $H_2^{s'}$ is contained in the sup-norm Sobolev space H_∞^s and in other L_∞-type Sobolev spaces. Therefore the optimal rates of convergence given here are upper bounds for convergence rates in all H_∞^s spaces, no matter how smooth.

5. Extensions to Besov Spaces

Embedding theorems for Besov and Sobolev spaces can be used to extend the Sobolev results in Section 4 to sup-norm convergence rates for wavelet expansions of functions in Besov spaces.

The basic questions regarding embedding take the form: Given a fixed Besov space $B_p^{s,q}$ where s is the smoothness parameter, is it true that every Sobolev space H_2^r is contained in this space for sufficiently large r? Conversely given a Sobolev space H_2^s and fixed p, q, is it true that $B_p^{r,q}$ is contained in this space for sufficiently large r?

The answer is yes and follows from Sobolev embedding theorems. For such embedding theorems see [BL], [De]. We now state only those embedding results for $\mathbb{R}^d$ that we need.

Theorem 5.1: *For* $s \in \mathbb{R}$ *and* $1 \leq p \leq \infty$

$$B_p^{s,1} \subset H_p^s \subset B_p^{s,\infty} \quad \text{on } \mathbb{R}^d.$$

In particular, $B_2^{s,1} \subset H_2^s \subset B_2^{s,\infty}$*.*

Combined with Theorem 4.1, this yields the following (best order of approximation is defined in Definition 2.3).

Theorem 5.2: *Given a multiresolution approximation* $\{P_n\}$*, let* r *be the best order of approximation of* P_n *in* H_2^s*, as given in Theorem 4.5. Then the order of approximation in the Besov space* $B_2^{s,1}$ *is at least* r*, while in* $B_2^{s,\infty}$ *it is at most* r*.*

In order to obtain results on optimal orders of convergence we need the following.

Lemma 5.3: *Let $2 \le p \le \infty$, and $q \ge 1$. Given $s \in \mathbb{R}$, for sufficiently large s_1, we have*

$$H_2^{s_1} \subset B_p^{s,q}. \tag{5.1a}$$

Similarly, if $1 \le p \le 2$, then for sufficiently large s_1,

$$B_p^{s_1,q} \subset H_2^s. \tag{5.1b}$$

In order to obtain exact optimal sup-norm orders of convergence for functions in Besov spaces we compare the scales $\{H_2^s\}_s$ and $\{B_p^{sq}\}_s$, for fixed p and q. To this end, note that by the above inclusions we have:

Corollary 5.4: *For any $1 \le q \le \infty$, the scales of spaces $\{H_2^s\}_{s\ge 0}$ and $\{B_2^{s,q}\}_{s\ge 0}$ are intertwined; that is, for any fixed s and sufficiently large s_1, $H_2^{s_1} \subset B_2^{s,q}$ and $B_2^{s_1,q} \subset H_2^s$.*

The next corollary allows us to find precise optimal convergence rates in the scale $\{B_2^{s,q}\}_{s\ge 0}$ for any fixed q. Since this scale intertwines with the scale $\{H_2^s\}_{s\ge 0}$, the optimal rate must be the same, namely:

Corollary 5.5: *If the best Sobolev parameter $\sigma \ne 0$, then in the scale $\{B_2^{s,q}\}_{s\ge 0}$ of L_2-Besov spaces, the wavelet collection ψ^λ [or scaling function ϕ] has optimal pointwise order of approximation given by*
(i) $\sigma - d/2$ if $K_{\sigma+1/2}(c) = O(1/c)$, and
(ii) $(\sigma - d/2)^-$ otherwise.
(Here K can be replaced by I or M).

6. Pointwise Convergence in Uniformly Local Sobolev Spaces

In this section we will see that the diagram in the figure in Section 4. above also applies to expansions of functions in uniformly local spaces $H_{2,ul}^s$, when the decay rate t of the scaling function satisfies $t - d \ge s$ (Theorem 6.2 below). In addition, on compact subsets the rates in the diagram apply to functions in local Sobolev spaces $H_{2,loc}^s$, when the wavelet has compact support.

The study of convergence properties in local rather than global Sobolev spaces is a naturally motivated pursuit, given that pointwise convergence properties should be determined strictly locally for compact wavelets, and essentially locally (i.e., with small modifications) for wavelets with rapid decay.

Definitions 6.1: The *decay rate* of a function ϕ is

$$\sup\{t : |\phi(x)| \le K|x|^{-t} \text{ for some } K > 0\}.$$

We will assume here our decay rates t are positive unless otherwise specified.

The *local Sobolev space* $H^s_{2,loc}$ is $\{f \in L_2(\mathbb{R}^d) : f\eta \in H^s_2 \ \forall\ \eta \in C_0^\infty\}$, where C_0^∞ is compactly supported C^∞ functions.

The *uniformly local Sobolev space* $H^s_{2,ul}$ is $\{f \in L_2(\mathbb{R}^d) : \|f\|^{ul}_s < \infty\}$, where the uniformly local norm $\|\cdot\|^{ul}_s$ is defined by:

$$\|f\|^{ul}_s \equiv \sup_{x\in\mathbb{R}^d} \|f\|_{s,B_x},$$

with B_x is the unit ball centered at x, where the local norm is defined by

$$\|f\|_{s,B_x} \equiv \inf_{f^*\big|_{B_x}=f,\ f^*\in H^s_2} \|f^*\|_{H^s_2}. \tag{6.1}$$

Similarly, the space $H^s_p \equiv \{f \in L^p : (1-\Delta)^{s/2} f \in L^p\}$ has a local version $H^p_{s,ul}$ defined analogously to the above with the norm $\|f^*\|_{H^s_2}$ in (6.1) replaced by the norm of the Sobolev space H^s_p.

As their name implies, these spaces are local versions of the global Sobolev spaces H^s_p. By definition, $f(x)$ is in the purely local space $H^p_{s,loc}$ if the product $f\eta \in H^s_p$ for any compactly supported smooth function $\eta(x)$. Membership in a uniformly local space, on the other hand, requires that the size of $f\eta$ be uniformly bounded if $\eta(x)$ is translated to $\eta(x-a)$ for any $a \in \mathbb{R}^d$. Thus uniformly local spaces are not entirely local because of global sup-norm constraints on the local norms.

An important property of the scale of uniformly local spaces is that there are bi-directional inclusions among the spaces $H^s_{p,ul}$ for different values of p. Thus, for example, for any p and q (including ∞), and for sufficiently large r, the space $H^r_{p,ul}$ is contained in the space $H^s_{q,ul}$. More interestingly, this type of inclusion also includes H^s_∞ and related standard spaces of functions with bounded derivatives.

This is useful in the following way. First, if $\phi(x)$ has decay rate t (Def. 6.1) with $t - d \geq \sigma - d/2$ (which holds, for example, for all r-regular wavelets [Me]), then Corollary 4.6 implies the optimal convergence rate in any scale of uniformly local spaces $H^p_{s,ul}$ (including $p = \infty$) is either $\sigma - d/2$ or $(\sigma - d/2)^-$, i.e., the same as in the Corollary. Now this result for the spaces $H^r_{p,ul}$ can be broadened to more general scales of smoothness spaces using the above inclusions. This includes spaces of functions defined by L_∞ bounds on their derivatives. The caveat, however, is that the scaling function ϕ must have the above sufficiently rapid decay.

With this motivation, we present convergence rate results for uniformly local Sobolev spaces. The results below are for the spaces $H^s_{2,ul}$ and can be extended to other scales of spaces via the above-mentioned inclusions. The results for $H^s_{2,ul}$ are essentially local versions of the above rates of convergence results, modulo the spatial uniformity assumptions on functions in $H^s_{2,ul}$. However, the uniformity conditions are not restrictive; for example, similar uniformity assumptions also hold, e.g., for L_∞ Sobolev spaces. We assume our working spaces have uniformly bounded local

L_2 Sobolev norms rather than L_∞ Sobolev norms, since the latter are more difficult to work with in the present context. Our results will also directly extend to local Sobolev spaces $H^s_{2,loc}$ with some caveats.

Recall from the definitions that approximation order 0 in a space X means the error $(I - P_n)f$ fails to have any positive rate of decay for some $f \in X$. Proofs of Theorem 6.2 and Corollary 6.3 will appear in [KR2].

Theorem 6.2 (Localization): *The multiresolution or wavelet expansion (2.3) corresponding to a scaling function $\phi \in [RB]$ has a best pointwise approximation order of at least $min(r, t-d)$ in $H^s_{2,ul}$, with r the rate of best approximation in H^s_2 and $t > d$ the decay rate of ϕ.*

The proofs of these uniform convergence results for uniformly local Sobolev spaces use embedding theorems that reduce the proof to the case $p = 2$ and write f in $H^s_{2,ul}(\mathbb{R}^d)$ as a decomposition of its local and global parts.

Corollary 6.3: *If the best Sobolev parameter $\sigma \le t - d/2$ (where t is the decay rate of ϕ), then:*
(a) The optimal approximation order in the scale of spaces $H^s_{2,ul}$ is exactly $\sigma - d/2$ if $I_{\sigma+1/2}(c) = O(1/c)$ [where I can be replaced by K or M], and $(\sigma - d/2)^-$ otherwise.
(b) The same exact optimal approximation order holds in the scale of uniformly local spaces $H^s_{p,ul}$ for fixed $1 \le p \le \infty$, and in particular also in the scale $H^s_{\infty,ul}$ and thus H^s_∞.

Proposition 6.4: *If ϕ is compactly supported, the best pointwise approximation rate for the expansion of any $f \in H^s_{2,loc}$ on any compact $K \subset \mathbb{R}^d$ is the same as the rate for the global space H^s_2.*

7. Examples:

To illustrate these results we briefly mention applications to some well-known wavelet approximations. Details for these rates of convergences for $H^s_2(\mathbb{R}^1)$ will appear in [KR2].

7.1 Haar wavelets

By Theorem 4.5, Haar expansions in H^s_2 have best order of convergence

$$r = \begin{cases} 0, & s \le 1/2 \\ s - 1/2, & 1/2 < s < 3/2, \\ 1^-, & s = 3/2 \\ 1, & s > 3/2 \end{cases} . \tag{7.1}$$

By Corollary 4.6, the optimal approximation order for such expansions (i.e., for arbitrarily smooth functions) is 1.

The optimal order in scale of Besov spaces B^{sq}_2 is the same as the optimal order in the scale of L_2 Sobolev spaces H^s_2.

By Theorem 6.2 these same orders also hold in the uniformly local Sobolev spaces H^s_{ul}. Since ϕ is compactly supported, these orders of convergence for $f(x)$ hold uniformly on compact subsets of $\mathbb{R}^d$, for any $f \in H^s_{2,loc}$ by Proposition 6.4. By Corollary 6.3 this optimal order also holds, among others, in the scale H^s_∞ of L_∞ Sobolev spacess.

7.2 Meyer wavelets

In the case of Meyer wavelets, $\widehat{\phi} \in C_0^\infty$ and $\sigma = \infty^-$. So we have order of convergence $s - 1/2$ in *every* Sobolev space H^s_2, $s > 1/2$, with a convergence order of 0 for $s \leq 1/2$. Thus $f \in \cap_s H^s$ in the intersection of all Sobolev spaces, the convergence is faster than any finite order r.

If $p = 2$, the optimal order in scale of Besov spaces is the same as the optimal order in the scale of Sobolev spaces.

By Theorem 6.2 the same holds in $H^s_{2,ul}$. Thus the optimal order of convergence is ∞, i.e., convergence rates have no wavelet-based limitations for very smooth f.

7.3 Battle Lemarié wavelets

By Theorem 4.5, Battle-Lemarié expansions (as well as order one spline expansions - note the scaling spaces V_j are the same) have order of convergence

$$r = \begin{cases} 0, & s \leq 1/2, \\ s - 1/2, & 1/2 < s < 5/2 \\ 2^-, & s = 5/2 \\ 2, & s > 5/2 \end{cases}$$

in H^s_2.

Again, if $p = 2$, the optimal order in scale of Besov spaces is the same as the optimal order in the scale of Sobolev spaces.

In the uniform local spaces $H^s_{2,ul}$ the same approximation rates again hold. Analogous results hold for the higher order versions of these spline wavelets, as well as the corresponding spline expansions.

7.4 Daubechies wavelets

For standard Daubechies wavelets of order 2, by Theorem 4.5 for H^s_2, the best order of approximation is

$$r = \begin{cases} 0, & s \leq 1/2 \\ s - 1/2, & 1/2 < s < 5/2 \\ 2^-, & s = 5/2 \\ 2, & s > 5/2 \end{cases}.$$

Similar analyses can be done for higher order Daubechies expansions.

As above, if $p = 2$, the optimal order in the scale of Besov spaces is the same as the optimal order in the scale of Sobolev spaces.

As before the global space H_2^s can be replaced by the uniformly local space $H_{2,ul}^s$. The optimal order of convergence for these Daubechies wavelets is 2. Since the wavelets are compactly supported, these are entirely local results, and for any $f \in H_{2,loc}^s$, these exact approximation rates hold uniformly on any compact $K \subset \mathbb{R}^d$.

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