To Appear in the JOURNAL OF TIME SERIES ANALYSIS, 2002

Deconvolution of fractional Brownian motion *†‡

Vladas Pipiras and Murad S. Taqqu
Boston University

April 2, 2001

Abstract

We show that a fractional Brownian motion with index $H' \in (0,1)$ can be represented as an explicit transformation of a fractional Brownian motion with index $H \in (0,1)$. In particular, when $H' = 1/2$ we obtain a deconvolution formula (or autoregressive representation) for fractional Brownian motion. We work both in the "time domain" and the "spectral domain" and contrast the advantages of one domain over the other.

1 Introduction

A stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ is self-similar with index $H > 0$ (H-ss) if, for any $a > 0$,

$$\{X(at)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{a^H X(t)\}_{t\in\mathbb{R}},$$

where $\stackrel{d}{=}$ stands for the equality in the sense of the finite-dimensional distributions. An H-ss process which is Gaussian and has stationary increments is called a fractional Brownian motion (fBm). The self-similarity index H then lies in the interval $(0,1)$ and a classical notation for fBm is $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$. When $H = 1/2$, the fBm $B_{1/2}$ is the usual Brownian motion (Bm). The fBm is called standard if $EB_H^2(1) = 1$.

FBm is a continuous time series which is typically characterized by a moving-average representation. We will show that this representation can be inverted and that, consequently, fBm also admits an autoregressive representation. We show, more generally, that it is possible to represent a fBm with a given index H' in terms of a fBm involving an index $H \neq H'$.

It is well-known (see Samorodnitsky and Taqqu [9], p. 321) that there is an integral transformation of Bm which provides a representation for fBm with index $H \in (0,1)$. It is the moving average representation of fBm and is given by

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_{\mathbb{R}} f_H(t,u)dB_{1/2}(u)\right\}_{t\in\mathbb{R}}, \tag{1.1}$$

*This research was partially supported by the NSF Grant ANI-9805623 at Boston University.

†*AMS Subject classification.* Primary 60G18; secondary 60H05, 26A33.

‡*Keywords and phrases:* fractional Brownian motion, fractional integral and derivative, integral with respect to fractional Brownian motion, autoregressive representation.

where B_H is a standard fBm with index H ($H \neq 1/2$), $B_{1/2}$ is a standard Bm and where

$$f_H(t,u) = \frac{1}{C_1(H)} \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right) \tag{1.2}$$

and

$$C_1(H) = \left\{ \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right)^2 du + \frac{1}{2H} \right\}^{1/2},$$

with $a_+^{H-\frac{1}{2}} = a^{H-\frac{1}{2}}$, if $a > 0$, and $a_+^{H-\frac{1}{2}} = 0$, if $a \leq 0$.

Suppose now that we start with a fBm instead. Is there an integral transformation of it which leads back to Bm? Or, more generally, is there a transformation which provides a representation for fBm with a different self-similarity index? The answer to both questions is given in the following theorem.

Theorem 1.1 *Let B_H and $B_{H'}$ be two standard fBm's with indices $H \in (0,1)$ and $H' \in (0,1)$, respectively, and suppose that $H \neq H'$. Then*

$$\{B_{H'}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} f_{H,H'}(t,u) dB_H(u) \right\}_{t\in\mathbb{R}}, \tag{1.3}$$

where

$$f_{H,H'}(t,u) = \frac{C_1(H)\Gamma(H'+1/2)}{C_1(H')\Gamma(H+1/2)} \frac{1}{\Gamma(H'-H+1)} \left((t-u)_+^{H'-H} - (-u)_+^{H'-H} \right) \tag{1.4}$$

(with $C_1(1/2) = 1$).

A representation similar to (1.3) was obtained by Barton and Poor ([1], p. 958). Their result, however, does not cover Theorem 1.1 because the integral $\int_{\mathbb{R}} f(u) dB_H(u)$ in Barton and Poor [1] is defined for a smaller class of integrands f ($f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, bounded and continuous) and, moreover, for $H \in (1/2, 1)$ only.

By setting $H = 1/2$ in the representation (1.3) one recovers the moving average representation (1.1). When $H \neq 1/2$, however, the representation (1.3) is significantly different from (1.1). This is because the process $B_H(u)$ does not have independent increments and hence the measure corresponding to $dB_H(u)$ is not independently scattered.

We will provide in the sequel two perspectives on the integral representation (1.3): a "time domain" perspective and a "spectral domain" one. The difference between the two perspectives is that in the "spectral domain" we think in terms of the Fourier transforms. For the kernel function $f_{H,H'}(t,\cdot)$ to have the $L^2(\mathbb{R})$-Fourier transform (that is, to be in $L^2(\mathbb{R})$) we suppose, in the "spectral domain", that $|H - H'| < 1/2$. In this sense, the "time domain" perspective is superior to that of the "spectral domain". Theorem 1.1 is proved in the "time domain" in Section 2. We consider the "spectral domain" in Section 3.

By setting $H' = 1/2$ in Theorem 1.1, we obtain a *deconvolution formula* or *autoregressive representation* for fBm.

Corollary 1.1 *Let B_H be a standard fBm with index $H \in (0,1)$ ($H \neq 1/2$) and let $B_{1/2}$ be a standard Bm. Then*

$$\{B_{1/2}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} f_{H,1/2}(t,u) dB_H(u) \right\}_{t\in\mathbb{R}}, \tag{1.5}$$

where

$$f_{H,1/2}(t,u) = \frac{C_1(H)}{\Gamma(H+1/2)\Gamma(3/2-H)} \left((t-u)_+^{\frac{1}{2}-H} - (-u)_+^{\frac{1}{2}-H} \right). \tag{1.6}$$

The representation (1.5) should not be viewed as a way to *represent* the Brownian motion process $\{B_{1/2}(t)\}_{t\in\mathbb{R}}$ in terms of a more complicated one, namely fBm $\{B_H(t)\}_{t\in\mathbb{R}}$. Rather, it should be regarded as a way to *transform* the complicated fBm process $\{B_H(t)\}_{t\in\mathbb{R}}$ into the simpler one $\{B_{1/2}(t)\}_{t\in\mathbb{R}}$. This transformation has potential use in time series analysis. Dealing with uncorrelated random variables is much easier than dealing with correlated ones, particularly when the dependence is strong. Corollary 1.1 shows that such a transformation can indeed be achieved and it provides the precise kernel. Another potential application is mathematical finance, where the arbitrage opportunities that stem from the long-range dependence (Liptser and Shiryayev [2], Rogers [7]) can be eliminated by such a transformation. One is then left with the inherent randomness of the underlying noise.

Financial applications, however, require equality of random variables and σ-fields. To obtain such equalities, observe that, for each $t \in \mathbb{R}$, the integral in

$$B_{1/2}(t) = \int_{\mathbb{R}} f_{H,1/2}(t,u)dB_H(u) \tag{1.7}$$

is a well-defined random variable because it is the $L^2(\Omega)$ limit of some discrete approximations. By Corollary 1.1, the resulting process $\{B_{1/2}(t)\}_{t\in\mathbb{R}}$ has the finite-dimensional distributions of standard Bm. The following proposition shows that for each $t \in \mathbb{R}$, the random variable $\int_{\mathbb{R}} f_{1/2,H}(t,u)dB_{1/2}(u)$ is a.s. equal to the original $B_H(t)$ and, moreover, for $t \geq 0$, the Brownian motion $B_{1/2}$ carries the same information as the original fBm B_H.

Proposition 1.1 *Suppose that $\{B_H(t)\}_{t\in\mathbb{R}}$ is a standard fBm with index $H \in (0,1)$ $(H \neq 1/2)$ and that* (1.7) *holds for $t \in \mathbb{R}$. Let $\mathcal{F}_{H,t}$ and $\mathcal{F}_{1/2,t}$ be the σ-algebras generated by B_H and $B_{1/2}$ up to time $t \in \mathbb{R}$, respectively. Then, the inversion formula*

$$B_H(t) = \int_{\mathbb{R}} f_{1/2,H}(t,u)dB_{1/2}(u) \tag{1.8}$$

holds for each $t \in \mathbb{R}$ almost everywhere and, consequently, $\mathcal{F}_{H,t} = \mathcal{F}_{1/2,t}$ up to sets of measure zero for each $t > 0$.

This proposition is proved in Section 4. The equality $\mathcal{F}_{H,t} = \mathcal{F}_{1/2,t}$ fails for $t < 0$ because the function $f_{H,H'}(t,u)$ has support on $(-\infty, 0)$ for $t < 0$. For $t > 0$, $f_{H,H'}(t,u)$ has support on $(-\infty, t)$. A similar proposition holds if $1/2$ is replaced by $H' \in (0,1)$.

For simplicity of notation, we will use in the sequel a more convenient parametrization of fBm. We let

$$\kappa = H - \frac{1}{2}$$

and denote a fBm $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ with index $H \in (0,1)$ by $B^\kappa = \{B^\kappa(t)\}_{t\in\mathbb{R}}$. Observe that $H \in (0,1)$ now corresponds to $\kappa \in (-1/2, 1/2)$. We will refer to the two parametrizations as to the H-parametrization and the κ-parametrization of fBm. (The index κ is usually denoted d in the statistical literature, but in this context the letter d may be confusing.)

The paper is structured as follows. In Section 2.1 we recall the definitions of some fractional integral operators which are related to fBm. Their semigroup property is the key to understanding Theorem 1.1. Section 2.2 deals with definition of the integral which appears on the right-hand side of (1.3). In Section 2.3 we use the framework developed in Sections 2.1 and 2.2 to prove Theorem 1.1. The structure of Section 3 is similar to that of Section 2 with the difference that we work in the "spectral domain" instead. In Section 4 we prove Proposition 1.1. Finally, we conclude the paper with alternative representations in Section 5.

2 "Time domain" perspective

2.1 Fractional integrals and fractional Brownian motion

Fractional integrals $I^{\alpha}_{\mp}\phi$ of order $\alpha > 0$ on the real line of a function ϕ are defined by

$$(I^{\alpha}_{\mp}\phi)(s) = \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}} \phi(u)(s-u)^{\alpha-1}_{\mp}du = \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}} \phi(u)(u-s)^{\alpha-1}_{\pm}du,\ s\in\mathbb{R} \tag{2.1}$$

(see Chapter 2 in Samko et al. [8] and, in particular, (5.2) and (5.3) on p. 94). While the first representation indicates that $I^{\alpha}_{\mp}\phi$ involves the convolution of ϕ with the function $u^{\alpha-1}_{\mp}$, the second representation is more convenient to manipulate. Observe also that $(I^{\alpha}_{+}\phi)(s)$ involves integration on $(-\infty, s)$ while $(I^{\alpha}_{-}\phi)(s)$ involves integration on (s,∞).

We will need to define the integral (2.1) for $-1 < \alpha < 1$ as well. To understand first informally how this can be done, suppose that $-1 < \alpha < 0$ and observe that (2.1) can be rewritten as $(\Gamma(-|\alpha|))^{-1}\int_0^\infty \phi(s\pm u)u^{\alpha-1}du$. For nontrivial ϕ, this integral diverges at $u = 0$ for every s, like $(\Gamma(-|\alpha|))^{-1}\int_0^\infty \phi(s)u^{\alpha-1}du$. Subtracting the divergent term yields

$$\frac{1}{\Gamma(-|\alpha|)}\int_0^\infty (\phi(s\pm u)-\phi(s))\,u^{\alpha-1}du = \frac{|\alpha|}{\Gamma(1-|\alpha|)}\int_0^\infty (\phi(s)-\phi(s\pm u))\,u^{-(|\alpha|+1)}du.$$

This way of proceeding can be made rigorous for $-1 < \alpha < 0$ by considering fractional Marchaud derivatives $\mathbf{D}^{\alpha}_{\mp}\phi$ on the real line of a function ϕ of order $0 < \alpha < 1$ (we will set below $I^{\alpha}_{\mp}\phi = \mathbf{D}^{-\alpha}_{\mp}\phi$, $-1 < \alpha < 0$). The derivatives $\mathbf{D}^{\alpha}_{\mp}\phi$ are defined by

$$\mathbf{D}^{\alpha}_{\mp}\phi = \lim_{\epsilon\to 0}\mathbf{D}^{\alpha}_{\mp,\epsilon}\phi,$$

where

$$(\mathbf{D}^{\alpha}_{\mp,\epsilon}\phi)(s) = \frac{\alpha}{\Gamma(1-\alpha)}\int_\epsilon^\infty \frac{\phi(s)-\phi(s\pm u)}{u^{1+\alpha}}du$$

(see Samko et al. [8], p. 111). The Marchaud fractional derivatives can thus be viewed as an extension of the fractional integral operators $I^{\alpha}_{\mp}$ with $\alpha > 0$ to $-1 < \alpha < 0$. The idea, as sketched above, is closely connected to the regularization of divergent integrals. One can show that for "sufficiently good" functions ϕ,

$$\mathbf{D}^{\alpha}_{\mp}\phi = \text{p.f. } I^{-\alpha}_{\mp}\phi,\ 0 < \alpha < 1,$$

where p.f. stands for the finite part of a divergent integral in the Hadamard sense (see Samko et al. [8], p. 113).

As indicated above, we use in the sequel the notation

$$I^{\alpha}_{\mp}\phi = \mathbf{D}^{-\alpha}_{\mp}\phi, \text{ for } -1 < \alpha < 0, \quad \text{and} \quad I^{0}_{\mp}\phi = \phi.$$

We also note that the operators I^{α}_{+} and I^{α}_{-} with $\alpha > -1$ are related by the formula

$$QI^{\alpha}_{\pm}\psi = I^{\alpha}_{\mp}Q\psi, \tag{2.2}$$

where $(Q\psi)(x) = \psi(-x), x \in \mathbb{R}$ (see (5.9) and (5.61) in Samko et al. [8], p. 95 and 111, after correcting the typo in the reference (5.61)).

The following lemma relates fBm and the fractional integral operators I^{α}_{-} introduced above: the kernel functions in the representations (1.1) and (1.3) can be expressed via these operators.

Lemma 2.1 *Let $\alpha > -1$, $\alpha \neq 0$, and $t \in \mathbb{R}$. Then*

$$(t-s)^{\alpha}_{+} - (-s)^{\alpha}_{+} = \Gamma(\alpha+1)(I^{\alpha}_{-}1_{[0,t)})(s), \tag{2.3}$$

where $1_{[0,t)} = -1_{[t,0)}$, if $t < 0$.

PROOF: Let $t > 0$. If $\alpha > 0$, then

$$(t-s)^{\alpha}_{+} - (-s)^{\alpha}_{+} = \alpha \int_{\mathbb{R}} 1_{[0,t)}(u)(u-s)^{\alpha-1}_{+}du = \Gamma(\alpha+1)(I^{\alpha}_{-}1_{[0,t)})(s).$$

If $-1 < \alpha < 0$, we have

$$\begin{aligned}\Gamma(\alpha+1)(\mathbf{D}^{-\alpha}_{-,\epsilon}1_{[0,t)})(s) &= (-\alpha)\int_{\epsilon}^{\infty}\Big(1_{[0,t)}(s) - 1_{[0,t)}(s+u)\Big)u^{\alpha-1}du \\ &= \alpha\int_{\mathbb{R}} 1_{[0,t)}(v)(v-s)^{\alpha-1}_{+}1_{\{\epsilon<v-s\}}(v)dv + 1_{[0,t)}(s)\epsilon^{\alpha}.\end{aligned} \tag{2.4}$$

One can verify that, if $\epsilon < t$, the function (2.4) equals $(t-s)^{\alpha}_{+} - (-s)^{\alpha}_{+}$, $s \in \mathbb{R}$, on the set $A_{\epsilon} = \{s < -\epsilon\} \cup \{0 \le s < t-\epsilon\} \cup \{s \ge t\}$. The relation (2.3) follows by letting $\epsilon \to 0$ (recall that in our notation $I^{\alpha}_{-}\phi = \mathbf{D}^{-\alpha}_{-}\phi$). When $t < 0$, the proof of (2.3) is similar. $\Box$

Fbm can thus be viewed as the integral of the kernel $I^{\kappa}_{-}1_{[0,t)}$ with respect to Bm. Indeed, by applying Lemma 2.1, we can reexpress the moving average representation (1.1) of fBm as

$$\{B^{\kappa}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\frac{\Gamma(\kappa+1)}{c_1(\kappa)}\int_{\mathbb{R}}(I^{\kappa}_{-}1_{[0,t)})(s)dB^{0}(s)\right\}_{t\in\mathbb{R}}, \ \kappa \in (-1/2, 1/2), \tag{2.5}$$

where $c_1(\kappa) = C_1\left(\kappa + \frac{1}{2}\right)$.

The following lemma will be used in the proof of Theorem 1.1. It states that the semigroup property of fractional integral operators

$$I^{\alpha}_{-}(I^{\beta}_{-}\phi) = I^{\alpha+\beta}_{-}\phi, \tag{2.6}$$

valid for $\alpha > 0$, $\beta > 0$ and "sufficiently good" functions ϕ (Samko et al. [8], p. 96), holds also for step functions ϕ when $\alpha > 0$ and $\beta > -1$.

Lemma 2.2 *Let $\alpha > 0$, $\beta > -1$ and $t \in \mathbb{R}$. Then*

$$I^{\alpha}_{-}(I^{\beta}_{-}1_{[0,t)}) = I^{\alpha+\beta}_{-}1_{[0,t)},$$

where $1_{[0,t)} = -1_{[t,0)}$, if $t < 0$.

PROOF: Observe that, if $\alpha > 0$ and $\beta > -1$, then

$$\int_{\mathbb{R}} \left((t-u)_+^{\beta} - (-u)_+^{\beta}\right)(u-s)_+^{\alpha-1}du = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\left((t-s)_+^{\alpha+\beta} - (-s)_+^{\alpha+\beta}\right) \tag{2.7}$$

(to get this, note that the integral on the LHS is on $u \in (s,t)$ and make the change of variables $u = s + (t-s)z$). Now, by (2.1) and (2.3),

$$I_-^{\alpha}(I_-^{\beta}1_{[0,t)})(s) = \frac{1}{\Gamma(\alpha)\Gamma(\beta+1)}\int_{\mathbb{R}}\left((t-u)_+^{\beta} - (-u)_+^{\beta}\right)(u-s)_+^{\alpha-1}du$$

and hence, by (2.7) and (2.3),

$$I_-^{\alpha}(I_-^{\beta}1_{[0,t)})(s) = \frac{1}{\Gamma(\alpha+\beta+1)}\left((t-s)_+^{\alpha+\beta} - (-s)_+^{\alpha+\beta}\right) = (I_-^{\alpha+\beta}1_{[0,t)})(s).$$

□

2.2 Integration with respect to fractional Brownian motion in the "time domain"

The result (1.3) of Theorem 1.1 involves the integral of a deterministic function with respect to fBm. We recall here some basic facts, established in Pipiras and Taqqu [4], involving these type of integrals.

Suppose that B^{κ} is a standard fBm with index $\kappa \in (-1/2, 1/2)$. The integral $\int_{\mathbb{R}} f(u)dB^{\kappa}(u)$ can be defined in the $L^2(\Omega)$-sense, as the limit of some discrete approximations, for the following classes of integrands f:

$$\Lambda^{\kappa} = \left\{ f : \int_{\mathbb{R}} [(I_-^{\kappa}f)(s)]^2 ds < \infty \right\}, \tag{2.8}$$

if $\kappa \in [0, 1/2)$, and

$$\Lambda^{\kappa} = \left\{ f : \exists\ \phi_f \in L^2(\mathbb{R}) \text{ such that } f = I_-^{-\kappa}\phi_f \right\}, \tag{2.9}$$

if $\kappa \in (-1/2, 0)$. The rationale behind the definition of Λ^{κ} when $\kappa \in (-1/2, 0)$ is that, by Theorem 6.1 in Samko et al. [8], p. 125, we have, for $f \in \Lambda^{\kappa}$,

$$\int_{\mathbb{R}} [(I_-^{\kappa}f)(s)]^2 ds = \int_{\mathbb{R}} [(I_-^{\kappa}(I_-^{-\kappa}\phi_f))(s)]^2 ds = \int_{\mathbb{R}} \phi_f^2(s)ds < \infty.$$

If $f \in \Lambda^{\kappa}$ with $\kappa \in (-1/2, 1/2)$, we shall say that the integral $\int_{\mathbb{R}} f(u)dB^{\kappa}(u)$ is well-defined in the "time domain".

The integral $\int_{\mathbb{R}} f(u)dB^{\kappa}(u)$ is an element of the Hilbert space $\overline{\mathrm{sp}}(B^{\kappa})$, which is the closure in $L^2(\Omega)$ of the span of all the increments of fBm B^{κ}. It is thus a Gaussian zero mean random variable and its variance is such that

$$\int_{\mathbb{R}} f(u)dB^{\kappa}(u) \stackrel{d}{=} \frac{\Gamma(\kappa+1)}{c_1(\kappa)}\int_{\mathbb{R}} (I_-^{\kappa}f)(s)dB^0(s) \tag{2.10}$$

(see Sections 3.2 and 3.3 in Pipiras and Taqqu [4]).

2.3 Proof of the main result in the "time domain"

In this section we prove Theorem 1.1. Since now we use the κ-parametrization of fBm, we restate the theorem in terms of this parametrization. We also express the kernel function in terms of the operators introduced earlier.

Theorem 2.1 *Let B^κ and $B^{\kappa'}$ be two standard fractional Brownian motions with indices $\kappa \in (-1/2, 1/2)$ and $\kappa' \in (-1/2, 1/2)$, respectively, and suppose that $\kappa \neq \kappa'$. Then*

$$\{B^{\kappa'}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_{\mathbb{R}} g_{t,\kappa,\kappa'}(u)dB^\kappa(u)\right\}_{t\in\mathbb{R}}, \tag{2.11}$$

where

$$\begin{aligned} g_{t,\kappa,\kappa'}(u) &= \frac{c_1(\kappa)\Gamma(\kappa'+1)}{c_1(\kappa')\Gamma(\kappa+1)}(I_-^{\kappa'-\kappa}1_{[0,t)})(u) && (2.12)\\ &= \frac{c_1(\kappa)\Gamma(\kappa'+1)}{c_1(\kappa')\Gamma(\kappa+1)}\frac{1}{\Gamma(\kappa'-\kappa+1)}\left((t-u)_+^{\kappa'-\kappa} - (-u)_+^{\kappa'-\kappa}\right). && (2.13)\end{aligned}$$

PROOF: The functions (2.12) and (2.13) are equal by Lemma 2.1. To prove (2.11), suppose first that $\kappa \in [0, 1/2)$. By Lemma 2.2, $I_-^\kappa(I_-^{\kappa'-\kappa}1_{[0,t)}) = I_-^{\kappa'}1_{[0,t)}$ and, in view of (2.5), for $\kappa' \in (-1/2, 1/2)$, $I_-^{\kappa'}1_{[0,t)}$ is the kernel of fBm and hence $I_-^{\kappa'}1_{[0,t)} \in L^2(\mathbb{R})$. Relation (2.8) then implies that $I_-^{\kappa'-\kappa}1_{[0,t)} \in \Lambda^\kappa$ and, by (2.10), the integral on the right-hand side of (2.11) is well-defined in the "time domain" and the corresponding process has the distribution

$$\begin{aligned} &\left\{\frac{\Gamma(\kappa+1)}{c_1(\kappa)}\int_{\mathbb{R}}\frac{c_1(\kappa)\Gamma(\kappa'+1)}{c_1(\kappa')\Gamma(\kappa+1)}I_-^\kappa(I_-^{\kappa'-\kappa}1_{[0,t)})(s)dB^0(s)\right\}_{t\in\mathbb{R}}\\ &\quad = \left\{\frac{\Gamma(\kappa'+1)}{c_1(\kappa')}\int_{\mathbb{R}}(I_-^{\kappa'}1_{[0,t)})(s)dB^0(s)\right\}_{t\in\mathbb{R}},\end{aligned}$$

which, by (2.5), is also the distribution of the left-hand side of (2.11).

Suppose now that $\kappa \in (-1/2, 0)$. By Lemma 2.2, $I_-^{\kappa'-\kappa}1_{[0,t)} = I_-^{-\kappa}(I_-^{\kappa'}1_{[0,t)})$ and $I_-^{\kappa'}1_{[0,t)} \in L^2(\mathbb{R})$, since $\kappa' \in (-1/2, 1/2)$. Hence by (2.9), $I_-^{\kappa'-\kappa}1_{[0,t)} \in \Lambda^\kappa$ and $I_-^\kappa(I_-^{\kappa'-\kappa}1_{[0,t)}) = I_-^\kappa(I_-^{-\kappa}(I_-^{\kappa'}1_{[0,t)})) = I_-^{\kappa'}1_{[0,t)}$. (The last equality follows from Theorem 6.1 in Samko et al. [8], p. 125, applied to the function $I_-^{\kappa'}1_{[0,t)}$.) The conclusion then follows as in the case $\kappa \in [0, 1/2)$. □

3 "Spectral domain" perspective

It is a part of the folklore, that it is easier to work in the "spectral domain" than in the "time domain" because convolutions in (2.11) become products in the "spectral domain". We shall provide a "spectral domain" proof of the representation (2.11). We show, however, below that the integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ can be defined in the "spectral domain" as long as f has the $L^2(\mathbb{R})$-Fourier transform $\widehat{f}(x) = \int_{\mathbb{R}} e^{iux}f(u)du$ and the function $|\widehat{f}(x)||x|^{-\kappa}$ is square integrable on the real line. Since the function $g_{t,\kappa,\kappa'}$, which appears on the right-hand side of (2.11) belongs to $L^2(\mathbb{R})$ if $|\kappa - \kappa'| < 1/2$, we pay a price by working in the "spectral domain": we have to restrict the range of (κ, κ').

To compute the Fourier transform $\widehat{g}_{t,\kappa,\kappa'}$, we first recall some facts on the Fourier transform of fractional integrals. It is a part of the folklore to write

$$\widehat{I^{\alpha}_{-}\phi} = \widehat{\phi}(x)/(ix)^{\alpha},\ \alpha > -1, \tag{3.1}$$

where

$$(ix)^{\alpha} = e^{\alpha \ln |x| + \mathrm{sign}(x)\frac{\alpha\pi i}{2}} = |x|^{\alpha} e^{\mathrm{sign}(x)\frac{\alpha\pi i}{2}}.$$

Observe that the semigroup property (2.6) can be heuristically verified if we use (3.1) to take the Fourier transforms of the left-hand side of (2.6) to obtain the Fourier transform of the right-hand side of (2.6). The issue is, for what type of functions does (3.1) hold.

The following lemma shows that the relation (3.1) holds for step functions ϕ when $\alpha \in (-1/2, 1/2)$. It will will be used to prove Theorem 1.1 in the "spectral domain".

Lemma 3.1 *Let $-1/2 < \alpha < 1/2$ and $t \in \mathbb{R}$. Then $I^{\alpha}_{-}1_{[0,t)} \in L^2(\mathbb{R})$ and its L^2-Fourier transform is given by*

$$\widehat{I^{\alpha}_{-}1}_{[0,t)}(x) = \widehat{1}_{[0,t)}(x)/(ix)^{\alpha}. \tag{3.2}$$

PROOF: The function $I^{\alpha}_{-}1_{[0,t)}$ is in $L^2(\mathbb{R})$ by using Lemma 2.1. Moreover, since $\alpha \in (-1/2, 1/2)$, we have

$$\int_{\mathbb{R}} |\widehat{1}_{[0,t)}(x)|^2 |x|^{-2\alpha} dx = \int_{\mathbb{R}} \left| \frac{e^{ixt} - 1}{ix} \right|^2 |x|^{-2\alpha} dx < \infty.$$

The result (3.2) then follows from Proposition 3.3, (3), in Pipiras and Taqqu [4]. □

Along with the moving average representation (2.5) there is also the spectral representation of a standard fBm (see Samorodnitsky and Taqqu [9], p. 328) given by

$$\{B^{\kappa}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \frac{1}{c_2(\kappa)} \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} |x|^{-\kappa} d\widetilde{B}(x) \right\}_{t\in\mathbb{R}}, \tag{3.3}$$

where $\kappa \in (-1/2, 1/2)$,

$$c_2(\kappa) = \left(\frac{2\pi}{\Gamma(2\kappa + 2)\sin(\pi(\kappa + 1/2))} \right)^{1/2}$$

and $\widetilde{B} = B^1 + iB^2$ is a complex Gaussian measure such that B^1 and B^2 are independent Gaussian random measures, independently scattered on $\mathbb{R}_+$ and satisfying $B^1(A) = B^1(-A)$, $B^2(A) = -B^2(-A)$ and $E(B^1(A))^2 = E(B^2(A))^2 = |A|/2$, for any Borel set A of a finite Lebesgue measure $|A|$ (see Section 7.2.2 in Samorodnitsky and Taqqu [9]). One can verify directly that the right-hand side of (3.3) is fBm by noting that it is a Gaussian $H = \kappa + \frac{1}{2}$ self-similar process with stationary increments. By using Lemma 3.1 we can reexpress (3.3) as follows.

Corollary 3.1 *Suppose that $B^{\kappa} = \{B^{\kappa}(t)\}_{t\in\mathbb{R}}$, $\kappa \in (-1/2, 1/2)$, is a standard fBm. Then*

$$\{B^{\kappa}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \frac{1}{c_2(\kappa)} \int_{\mathbb{R}} \widehat{I^{\kappa}_{-}1}_{[0,t)}(x) d\widetilde{B}(x) \right\}_{t\in\mathbb{R}}. \tag{3.4}$$

PROOF: The result follows from the spectral representation (3.3) and Lemma 3.1 by observing that the random measure corresponding to $e^{-\mathrm{sign}(x)\frac{\kappa\pi i}{2}}d\widetilde{B}(x)$ retains the properties of the measure corresponding to $d\widetilde{B}(x)$. $\square$

The following lemma, which will be used in the sequel, relates the normalization exponents $c_1(\kappa)$ and $c_2(\kappa)$.

Lemma 3.2 *If $\kappa \in (-1/2, 1/2)$, then*

$$\frac{\Gamma(\kappa+1)}{c_1(\kappa)} = \frac{(2\pi)^{1/2}}{c_2(\kappa)}. \tag{3.5}$$

PROOF: It follows from (3.4) that

$$E(B^\kappa(t))^2 = \frac{1}{(c_2(\kappa))^2}\int_{\mathbb{R}} |\widehat{I_-^\kappa 1}_{[0,t)}(x)|^2 dx. \tag{3.6}$$

On the other hand, by (2.5),

$$E(B^\kappa(t))^2 = \frac{(\Gamma(\kappa+1))^2}{(c_1(\kappa))^2}\int_{\mathbb{R}} [(I_-^\kappa 1_{[0,t)})(s)]^2 ds. \tag{3.7}$$

The relation (3.5) then follows from (3.6) and (3.7) by the Parseval's equality $2\pi\|f\|^2_{L^2(\mathbb{R})} = \|\hat{f}\|^2_{L^2(\mathbb{R})}$.
$\square$

The integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ can be also defined for a class of integrands f in the "spectral domain". This class, derived from the spectral representation of fBm (3.3), is

$$\widetilde{\Lambda}^\kappa = \left\{ f : f \in L^2(\mathbb{R}),\ \int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2\kappa} dx < \infty \right\},$$

where $\kappa \in (-1/2, 1/2)$. We will say that the integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ is well-defined in the "spectral domain" if $f \in \widetilde{\Lambda}^\kappa$ with $\kappa \in (-1/2, 1/2)$. The integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ is a Gaussian random variable belonging to the space $\overline{\mathrm{sp}}(B^\kappa)$ and is such that

$$\int_{\mathbb{R}} f(u)dB^\kappa(u) \stackrel{d}{=} \frac{1}{c_2(\kappa)}\int_{\mathbb{R}} \widehat{f}(x)|x|^{-\kappa} d\widetilde{B}(x). \tag{3.8}$$

The classes of integrands Λ^κ in the "time domain" (see Section 2.2) and $\widetilde{\Lambda}^\kappa$ in the "spectral domain" can be compared. It is shown in Pipiras and Taqqu [4] that $\widetilde{\Lambda}^\kappa$ is a strict subset of Λ^κ (unless $\kappa = 0$ in which case $\widetilde{\Lambda}^\kappa = \Lambda^\kappa = L^2(\mathbb{R})$) and, if $f \in \widetilde{\Lambda}^\kappa \subset \Lambda^\kappa$, then the integrals $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ in the "time domain" and the "spectral domain" are almost surely equal.

The following theorem is the "spectral domain" version of Theorem 2.1.

Theorem 3.1 *Suppose that the assumptions of Theorem 2.1 hold and that $|\kappa - \kappa'| < 1/2$. Then the integral on the right-hand side of (2.11) is well-defined in the "spectral domain" and the relation (2.11) holds.*

PROOF: If $|\kappa-\kappa'|<1/2$, then the function $g_{t,\kappa,\kappa'}$ in (2.12) belongs to $L^2(\mathbb{R})$ and, by Lemmas 3.1 and 3.2, its $L^2(\mathbb{R})$-Fourier transform is given by

$$\widehat{g}_{t,\kappa,\kappa'}(x)=\frac{c_2(\kappa)}{c_2(\kappa')}\widehat{1}_{[0,t)}(x)(ix)^{\kappa-\kappa'}=\frac{c_2(\kappa)}{c_2(\kappa')}\widehat{1}_{[0,t)}(x)|x|^{\kappa-\kappa'}e^{\mathrm{sign}(x)(\kappa'-\kappa)\pi i/2}.$$

Since

$$\int_{\mathbb{R}}|\widehat{1}_{[0,t)}(x)|^2|x|^{2(\kappa-\kappa')}|x|^{-2\kappa}dx=\int_{\mathbb{R}}\left|\frac{e^{ixt}-1}{ix}\right|^2|x|^{-2\kappa'}dx<\infty,$$

if $\kappa'\in(-1/2,1/2)$, the function $g_{t,\kappa,\kappa'}$ belongs to $\widetilde{\Lambda}^{\kappa}$. Hence, the integral on the right-hand side of (2.11) is well-defined in the "spectral domain" and, by (3.8), the corresponding process has the distribution

$$\left\{\frac{1}{c_2(\kappa)}\int_{\mathbb{R}}\frac{c_2(\kappa)}{c_2(\kappa')}\widehat{1}_{[0,t)}(x)|x|^{\kappa-\kappa'}|x|^{-\kappa}d\widetilde{B}'(x)\right\}_{t\in\mathbb{R}}$$

$$=\left\{\frac{1}{c_2(\kappa')}\int_{\mathbb{R}}\frac{e^{itx}-1}{ix}|x|^{-\kappa'}d\widetilde{B}'(x)\right\}_{t\in\mathbb{R}},\tag{3.9}$$

where $d\widetilde{B}'(x)=e^{\mathrm{sign}(x)(\kappa'-\kappa)\pi i/2}d\widetilde{B}(x)$. Since $d\widetilde{B}'(x)$ has the same properties as $d\widetilde{B}(x)$, the process (3.9) is just the spectral representation (3.3) of fBm $B^{\kappa'}$. □

4 The proof of Proposition 1.1

We shall prove Proposition 1.1 in the κ-parametrization of fBm. Let $\{B^{\kappa}(t)\}_{t\in\mathbb{R}}$ be a standard fBm with index $\kappa\in(-1/2,1/2)\setminus\{0\}$. Then, by Theorem 2.1, the process

$$B^0(t)=\int_{\mathbb{R}}g_{t,\kappa,0}(u)dB^{\kappa}(u),\ t\in\mathbb{R},\tag{4.1}$$

is a standard Bm. Let also $\mathcal{F}_t^{\kappa}$ and $\mathcal{F}_t^0$ be the σ-algebras generated by B^{κ} and B^0 up to time $t\in\mathbb{R}$, respectively. We want to show that

$$B^{\kappa}(t)=\int_{\mathbb{R}}g_{t,0,\kappa}(u)dB^0(u),\ t\in\mathbb{R},\tag{4.2}$$

which is the relation (1.8). Then, by (4.1) and (4.2), and since the function $g_{t,\kappa,\kappa'}$ in (2.13) has support on $(-\infty,t)$ for $t>0$, we would have $\mathcal{F}_t^{\kappa}=\mathcal{F}_t^0$ up to sets of measure zero for each $t>0$.

The proof of (4.2) for $t>0$ is in two steps. (We shall use some ideas from Norros, Valkeila and Virtamo [3].) Let $\widetilde{B}^{\kappa}(t)$ denote the right-hand side of (4.2). We shall first prove that

$$E(B^{\kappa}(t)|\mathcal{F}_t^0)=\widetilde{B}^{\kappa}(t),\ t>0,\tag{4.3}$$

and then establish $B^{\kappa}(t)=\widetilde{B}^{\kappa}(t)$ for $t>0$ a.e. To show (4.3), it is enough to prove that for any $s\le t$, $EB^{\kappa}(t)B^0(s)=E\widetilde{B}^{\kappa}(t)B^0(s)$ or, by (4.1),

$$E\left(B^{\kappa}(t)\int_{\mathbb{R}}g_{s,\kappa,0}(u)dB^{\kappa}(u)\right)=E\left(\int_{\mathbb{R}}g_{t,0,\kappa}(u)dB^0(u)B^0(s)\right).\tag{4.4}$$

By (2.10) and (2.12), the left-hand side of (4.4) becomes

$$\frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\frac{c_1(\kappa)}{\Gamma(\kappa+1)}\int_{\mathbb{R}}(I_-^\kappa 1_{[0,t)})(z)(I_-^\kappa I_-^{-\kappa}1_{[0,s)})(z)dz=\frac{\Gamma(\kappa+1)}{c_1(\kappa)}\int_{\mathbb{R}}(I_-^\kappa 1_{[0,t)})(z)1_{[0,s)}(z)dz,$$

which is obviously the right-hand side of (4.4). Since $EB^\kappa(t)^2 = E\widetilde{B}^\kappa(t)^2$ (both B^κ and $\widetilde{B}^\kappa$ are standard fBm's), we obtain that

$$E(B^\kappa(t)-\widetilde{B}^\kappa(t))^2 = EB^\kappa(t)(B^\kappa(t)-\widetilde{B}^\kappa(t)) - E\widetilde{B}^\kappa(t)(B^\kappa(t)-\widetilde{B}^\kappa(t))$$
$$= E\widetilde{B}^\kappa(t)(\widetilde{B}^\kappa(t)-B^\kappa(t)) - E\widetilde{B}^\kappa(t)(B^\kappa(t)-\widetilde{B}^\kappa(t)) = 2E\widetilde{B}^\kappa(t)(\widetilde{B}^\kappa(t)-B^\kappa(t))=0.$$

This last equality follows because $\widetilde{B}^\kappa(t)$ is $\mathcal{F}_t^0$-measurable and, by (4.3), $EB^\kappa(t)X = E\widetilde{B}^\kappa(t)X$ for all $\mathcal{F}_t^0$-measurable, square integrable random variables X. Hence $E(B^\kappa(t)-\widetilde{B}^\kappa(t))^2=0$, or $B^\kappa(t)=\widetilde{B}^\kappa(t)$ a.e.

Finally, to show (4.2) for $t<0$, one can replace $\mathcal{F}_t^0$ in (4.3) by $\mathcal{F}_0^0$ and follow the arguments of the case $t>0$.

5 Alternative representations

In this section, we consider two representations of fBm alternative to (1.3). The first representation concerns fBm on the real line while the second one deals with fBm on an interval.

Alternative representation on the real line. There are other kernels $g_{t,\kappa,\kappa'}$ in (2.11) which allow to go from one fBm B^κ to another fBm $B^{\kappa'}$. This is a consequence of the fact that fBm can be represented as an integral with respect to Bm in many equivalent ways. For example, one such representation, different from (1.1), is

$$\{B^\kappa(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\frac{1}{c_1(\kappa)}\int_{\mathbb{R}}\left((t-u)_-^\kappa-(-u)_-^\kappa\right)dB^0(u)\right\}_{t\in\mathbb{R}}. \tag{5.1}$$

Observe that, by the property (2.2) of fractional integrals and Lemma 2.1, $(t-u)_-^\kappa-(-u)_-^\kappa = -(I_+^\kappa 1_{(0,t]})(u)$. Consequently, by following the approach of Pipiras and Taqqu [4], we may define the integral with respect to fBm $\int f dB^\kappa$ as an element of the Hilbert space $\overline{\mathrm{sp}}(B^\kappa)$ which satisfies $\int f dB^\kappa =_d \Gamma(\kappa+1)c_1(\kappa)^{-1}\int I_+^\kappa f dB^0$ (compare with (2.10)). Then, by integrating the function $f = I_+^{\kappa'-\kappa}1_{(0,t]}$ with respect to fBm B^κ, we obtain as in the proof of Theorem 2.1, that

$$\int_{\mathbb{R}} f(u)dB^\kappa \stackrel{d}{=} \mathrm{const}\int_{\mathbb{R}}(I_+^\kappa(I_+^{\kappa'-\kappa}1_{(0,t]}))(s)dB^0(s) = \mathrm{const}\int_{\mathbb{R}}(I_+^{\kappa'}1_{(0,t]})(s)dB^0(s) \stackrel{d}{=} \mathrm{const}\, B^{\kappa'}(t).$$

After choosing a proper normalizing constant, we deduce that the representation (2.11) holds with the function

$$\frac{c_1(\kappa)\Gamma(\kappa'+1)}{c_1(\kappa')\Gamma(\kappa+1)}(I_+^{\kappa'-\kappa}1_{(0,t]})(u) = \frac{c_1(\kappa)\Gamma(\kappa'+1)}{c_1(\kappa')\Gamma(\kappa+1)}\frac{1}{\Gamma(\kappa'-\kappa+1)}\left((t-u)_-^{\kappa'-\kappa}-(-u)_-^{\kappa'-\kappa}\right) \tag{5.2}$$

for the kernel $g_{t,\kappa,\kappa'}(u)$. The use of the kernel (2.12)–(2.13), in contrast to (5.2), corresponds to the "causal" option since then a process at time t (with $t\ge 0$) is expressed as an integral over the past $(-\infty,t)$.

Representation on an interval. FBm B_H is represented in (1.1) as an integral with respect to Bm $B_{1/2}$ over the whole real line $\mathbb{R}$ and is defined at any time point $t \in \mathbb{R}$. If one is interested in fBm on an interval $[0,a]$, $a \geq 0$, then one can represent it with respect to Bm over $[0,a]$ only, as

$$\{B_H(t)\}_{t\in[0,a]} \stackrel{d}{=} \left\{ \int_0^t \tilde{f}_{1/2,H}(t,u)dB_{1/2}(u) \right\}_{t\in[0,a]},$$

where, for $0 \leq u \leq t \leq a$,

$$\tilde{f}_{1/2,H}(t,u) = \left(\frac{t}{u}\right)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) u^{\frac{1}{2}-H} \int_u^t s^{H-\frac{3}{2}}(s-u)^{H-\frac{1}{2}} ds \tag{5.3}$$

(see, for example, Norros et al. [3] or Pipiras and Taqqu [5]). It is then possible, by using the above representation, to relate two fBm's over an interval $[0,a]$. For example, a deconvolution integral for fBm would be

$$\{B_{1/2}(t)\}_{t\in[0,a]} \stackrel{d}{=} \left\{ \int_0^t \tilde{f}_{H,1/2}(t,u)dB_H(u) \right\}_{t\in[0,a]},$$

where, for $0 \leq u \leq t \leq a$,

$$\tilde{f}_{H,1/2}(t,u) = \left(H - \frac{1}{2}\right) u^{\frac{1}{2}-H} \int_u^t s^{H-\frac{3}{2}}(s-u)^{-H+\frac{1}{2}} ds - \left(\frac{t}{u}\right)^{H-\frac{1}{2}} (t-u)^{-H+\frac{1}{2}}.$$

We state these results without proofs. However, the present paper provides a methodology of how to show them. The idea is first to express the kernel (5.3) by using fractional integral operators and use this to define classes of integrands f along with integrals $\int_0^a f(u)dB_H(u)$ (as in Pipiras and Taqqu [5]). Then, the representations one seeks can be obtained, as it is done in this paper, by working with fractional integrals and derivatives (in particular, by using inverse fractional integral operators to get the necessary cancellations). This is illustrated in Pipiras and Taqqu [6].

References

[1] R. J. Barton and H. V. Poor. Signal detection in fractional Gaussian noise. *IEEE Transactions on Information Theory*, 34(5):943–955, 1988.

[2] R. S. Liptser and A. N. Shiryayev. *Theory of Martingales.* Kluwer Academic Publishers, Dordrecht, the Netherlands, 1989.

[3] I. Norros, E. Valkeila, and J. Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 15:571–587, 1999.

[4] V. Pipiras and M. S. Taqqu. Integration questions related to fractional Brownian motion. *Probability Theory and Related Fields*, 118(2):251–291, 2000.

[5] V. Pipiras and M. S. Taqqu. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli*, 2001. To appear.

[6] V. Pipiras and M. S. Taqqu. Fractional calculus and its connections to fractional Brownian motion. In P. Doukhan, G. Oppenheim, and M. S. Taqqu, editors, *Long-range Dependence: Theory and Applications.* Birkhäuser, 2001. Preprint.

[7] L. C. G. Rogers. Arbitrage with fractional Brownian motion. *Mathematical Finance*, 7:95–105, 1997.

[8] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives.* Gordon and Breach Science Publishers, 1993.

[9] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, New York, London, 1994.

Vladas Pipiras and Murad S. Taqqu
Department of Mathematics
Boston University
111 Cummington St.
Boston, MA 02215, USA
pipiras@math.bu.edu, murad@math.bu.edu