To Appear in BERNOULLI, 2002

# Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *†‡

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May 14, 2001

**Abstract**

Let $B_H$ be a fractional Brownian motion with self-similarity parameter $H \in (0,1)$ and $a > 0$ be a fixed real number. Consider the integral $\int_0^a f(u)dB_H(u)$, where $f$ belongs to a class of non-random integrands $\Lambda_{H,a}$. The integral will then be defined in the $L^2(\Omega)$-sense. One would like $\Lambda_{H,a}$ to be a *complete* inner product space. This corresponds to a desirable situation because then there is an isometry between $\Lambda_{H,a}$ and the closure of the span generated by $B_H(u)$, $0 \le u \le a$. We show in this work that, when $H \in (1/2,1)$, the classes of integrands $\Lambda_{H,a}$ one usually considers are *not complete* inner product spaces even though they are often assumed in the literature to be complete. Thus, they are isometric not to $\overline{\mathrm{sp}}\{B_H(u),\ 0 \le u \le a\}$ but only to a proper subspace. Consequently, there are (random) elements in that closure which cannot be represented by functions $f$ in $\Lambda_{H,a}$. We also show in contrast to the case $H \in (1/2,1)$ that there is a class of integrands for fractional Brownian motion $B_H$ with $H \in (0,1/2)$ on an interval $[0,a]$ which is a *complete* inner product space.

## 1 Introduction

A fractional Brownian motion (fBm) $B_H = \{B_H(u)\}_{u\in\mathbb{R}}$ with self-similarity parameter $H \in (0,1)$ is a zero mean Gaussian process which has stationary increments and is self-similar with index $H$ ($H$-ss), that is, for any $c > 0$,

$$\{B_H(cu)\}_{u\in\mathbb{R}} \stackrel{d}{=} \{c^H B_H(u)\}_{u\in\mathbb{R}},$$

*This research was partially supported by the NSF Grant ANI-9805623 at Boston University.

†*AMS Subject classification.* Primary 60H05, 60G18; secondary 26A33.

‡*Keywords and phrases:* fractional Brownian motion, integration in the $L^2$-sense, fractional integrals and derivatives, inner product spaces, completeness.

where $\stackrel{d}{=}$ stands for the equality in the sense of the finite-dimensional distributions. For notational simplicity we will use in the sequel another parametrization of fBm. Let

$$\kappa = H - \frac{1}{2},$$

so that the range $H \in (0,1)$ now corresponds to the range $\kappa \in (-1/2, 1/2)$. We will denote the fBm $B_H = \{B_H(u)\}_{u\in\mathbb{R}}$ in terms of the parameter $\kappa$ as $B^\kappa = \{B^\kappa(u)\}_{u\in\mathbb{R}}$. We will also assume that fBm $B^\kappa$ is standard, i.e. $E(B^\kappa(1))^2 = 1$. By using stationarity of the increments and $\kappa + \frac{1}{2}$–ss of a standard fBm $B^\kappa$, it is easy to see that the covariance function of $B^\kappa$ is given by

$$\Gamma^\kappa(u,v) = EB^\kappa(u)B^\kappa(v) = \frac{1}{2}\left\{|u|^{2\kappa+1} + |v|^{2\kappa+1} - |u-v|^{2\kappa+1}\right\},\ u,v \in \mathbb{R}. \tag{1.1}$$

When $\kappa = 0$, fBm $B^\kappa = B^0$ is the usual Brownian motion (Bm) which has independent increments. If $\kappa \neq 0$, the increments of fBm $B^\kappa$ are no longer independent: they are positively correlated if $\kappa \in (0,1/2)$, and negatively correlated if $\kappa \in (-1/2,0)$. Moreover, when $\kappa \in (0,1/2)$, the dependence at large time lags is so strong that the series $\sum_{k=1}^{\infty}\Gamma^\kappa(1,k)$ diverges. In this case, one says that fBm $B^\kappa$ exhibits long-range dependence. (For more information on fBm see, for example, Samorodnitsky and Taqqu (1994).)

In this work we deal with questions related to the $L^2(\Omega)$-integration of *deterministic* functions with respect to fBm $B^\kappa$ when $\kappa \in (-1/2,1/2)$. To define such integrals, one typically starts with an inner product space $(\mathcal{C}, (\cdot,\cdot)_{\mathcal{C}})$ of functions on a region of integration $R$ (say, $R = \mathbb{R}$ or $[0,a]$ with $a > 0$) such that $(1_{[0,s)}, 1_{[0,t)})_{\mathcal{C}} = EB^\kappa(s)B^\kappa(t)$ for all $s,t \in R$. Let $\overline{\mathrm{sp}}_R(B^\kappa)$ be the closure in $L^2(\Omega)$ of all possible linear combinations of the increments of fBm on $R$. If the map $1_{[0,t)} \mapsto B^\kappa(t)$ extends to the isometry between this class of functions $\mathcal{C}$ and the space $\overline{\mathrm{sp}}_R(B^\kappa)$, then the resulting isometry map is called the integral in the $L^2(\Omega)$–sense with respect to fBm of functions from $\mathcal{C}$.

The extension step is usually taken for granted (i.e. not proved). So, for example, when $\kappa \in (0,1/2)$ and the region of integration $R$ is $[0,a]$ with $a>0$, Carmona et al. (1999) and Kleptsyna et al. (1999) defined the class of integrands for fBm

$$\Lambda_a^\kappa = \left\{ f:[0,a]\mapsto \mathbb{R} \text{ such that } \int_0^a \left[ s^{-\kappa}\left(I_{a-}^\kappa u^\kappa f(u)\right)(s)\right]^2 ds < \infty \right\}, \tag{1.2}$$

where $I_{a-}^\kappa$ is a fractional integral operator defined in Section 2 below (see (2.1)). Another class of integrands, considered by Duncan et al. (2000), Kleptsyna et al. (1998) and Norros et al. (1999), is given by

$$|\Lambda|_a^\kappa = \left\{ f:[0,a]\mapsto \mathbb{R} \text{ such that } \int_0^a \int_0^a |f(u)||f(v)||u-v|^{2\kappa-1} du dv < \infty \right\}. \tag{1.3}$$

The classes of integrands $\Lambda_a^\kappa$ and $|\Lambda|_a^\kappa$ are assumed to be inner product spaces with the inner products

$$(f,g)_{\Lambda_a^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa}\int_0^a s^{-2\kappa}\left(I_{a-}^\kappa u^\kappa f(u)\right)(s)\left(I_{a-}^\kappa u^\kappa g(u)\right)(s) ds \tag{1.4}$$

($\Gamma(p)=\int_0^\infty e^{-v}v^{p-1}dv$, $p>0$, is the gamma function) and

$$(f,g)_{|\Lambda|_a^\kappa} = \kappa(2\kappa+1)\int_0^a\int_0^a f(u)g(v)|u-v|^{2\kappa-1} du dv, \tag{1.5}$$

respectively. (For details on the construction of classes of integrands $\Lambda_a^\kappa$ and $|\Lambda|_a^\kappa$ see Section 4 below.) All the authors claimed that both $\Lambda_a^\kappa$ and $|\Lambda|_a^\kappa$ are isometric to the Gaussian space $\overline{\text{sp}}_{[0,a]}(B^\kappa)$. Since $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ is a complete inner product space, both $\Lambda_a^\kappa$ and $|\Lambda|_a^\kappa$ have necessarily to be complete inner product spaces as well. We will show in this work that, when $\kappa\in(0,1/2)$, *neither the space of functions* $\Lambda_a^\kappa$ *nor the space of functions* $|\Lambda|_a^\kappa$ *is a complete inner product space.* Thus, they cannot be isometric to $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ itself: in fact, they are isometric only to proper linear subspaces of $\overline{\text{sp}}_{[0,a]}(B^\kappa)$. Consequently, there are (random) elements in $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ which can not be represented by functions $f$ belonging to either $\Lambda_a^\kappa$ or $|\Lambda|_a^\kappa$.

One area of applications of integrals with respect to fBm, where completeness of classes of integrands is relevant, involves prediction problems. Consider, for example, the problem of prediction the value of fBm at some future time $t > 0$ given its past from time 0 to time $a$ (with $a < t$), or in mathematical terms, to compute the conditional expectation $X = E(B^\kappa(t)|B^\kappa(s), s \in [0,a])$. It is well-known that $X \in \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. One would expect that $X = \int_0^a f dB^\kappa$. But, when $\kappa > 0$, in view of the above mentioned incompleteness results, there may be no $f \in \Lambda_a^\kappa$ or $|\Lambda|_a^\kappa$ such that $X = \int_0^a f dB^\kappa$. In fact, such an $f$ exists as is shown in Section 7 below, where the prediction problem for fBm is discussed.

Although $a > 0$ is assumed to be a real number, our results are also valid in the case $a = \infty$. (The space of integrands $|\Lambda|_a^\kappa$ with $a = \infty$ is considered in Duncan et al. (2000) and Norros et al. (1999).) As shown in Section 6 below, when $\kappa \in (0,1/2)$ incompleteness of classes of integrands in the case $a = \infty$ can be in principle deduced from that of classes of integrands in the case $a > 0$.

The results described above are in the spirit as those of Pipiras and Taqqu (2000), where the integration is over $\mathbb{R}$. The $\mathbb{R}$ setup, however, is quite different from that of $[0,a]$ considered herein. When $\kappa \in (0,1/2)$, an $\mathbb{R}$–analogue of the space of integrands $\Lambda_a^\kappa$ is the inner product space

$$\Lambda^\kappa = \left\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}} [(I_-^\kappa f)(s)]^2 ds < \infty \right\} \tag{1.6}$$

with the inner product

$$(f,g)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (I_-^\kappa f)(s)(I_-^\kappa g)(s)\ ds, \tag{1.7}$$

where $(I_-^\kappa f)(s) = (\Gamma(\kappa))^{-1} \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} du$, $s \in \mathbb{R}$, is a fractional integral on $\mathbb{R}$ and $c_1(\kappa)$ is some constant which depends on $\kappa$. As shown in Theorem 3.2 of Pipiras and Taqqu (2000), the inner product space $\Lambda^\kappa$ is not complete. We do not know whether one can use this to conclude immediately that $\Lambda_a^\kappa$

is incomplete. We shall provide a proof that relies indirectly on the incompleteness of $\Lambda^\kappa$. Obtaining results concerning integration over $[0,a]$ or $[0,\infty)$ is important in practice (most of the papers quoted above consider integration over this range).

Suppose now that the parameter $\kappa$ belongs to the range $(-1/2,0)$. (This range corresponds to $H \in (0,1/2)$ or no long-range dependence.) It is natural to ask whether there is a class of integrands for fBm $B^\kappa$ with $\kappa \in (-1/2,0)$ on an interval $[0,a]$ which is a complete inner product space. It turns out that such a class of integrands exists. It is an inner product space

$$\Lambda_a^\kappa = \left\{ f : \exists\ \phi_f \in L^2[0,a] \text{ such that } f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi_f(s))(u) \right\}, \tag{1.8}$$

with the inner product

$$(f,g)_{\Lambda_a^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa} \int_0^a s^{-2\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)(I_{a-}^\kappa u^\kappa g(u))(s)ds \tag{1.9}$$

$$= \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa} \int_0^a \phi_f(s)\phi_g(s)ds, \tag{1.10}$$

where $\phi_f, \phi_g \in L^2[0,a]$ are associated with the functions $f$ and $g$, respectively, by definition (1.8), and $I_{a-}^\kappa$ with $\kappa \in (-1/2,0)$ are fractional derivatives of order $(-\kappa)$. (The fractional derivatives $I_{a-}^\kappa$ with $\kappa \in (-1/2,0)$ are defined in Section 2 below. They satisfy the property $I_{a-}^\kappa I_{a-}^{-\kappa}\phi = \phi$ for any $\phi \in L^1[0,a]$ and $\kappa \in (-1/2,0)$ which explains the equality of (1.9) and (1.10).) For the construction of the class of integrands $\Lambda_a^\kappa$ with $\kappa \in (-1/2,0)$ and for the proof of its completeness see Section 4 below.

This work is organized as follows. In Section 2 we provide a quick review of fractional integrals and derivatives that are used in this work. Then, in Section 3, we represent fractional Brownian motion on an interval in terms of these fractional integrals and derivatives. We use this representation in Section 4 to construct the classes of integrands $\Lambda_a^\kappa$ with $\kappa \in (-1/2,1/2)$ and $|\Lambda|_a^\kappa$ with $\kappa \in (0,1/2)$. We also show, in Section 4, that $\Lambda_a^\kappa$ is a complete inner product space when $\kappa \in (-1/2, 0)$ and, in Section 5, that $\Lambda_a^\kappa$ and $|\Lambda|_a^\kappa$ are not complete inner product spaces when $\kappa \in (0, 1/2)$. In Section 6 we deal with the case $a = \infty$. We consider the prediction problem for fBm in Section 7. Finally, in Section 8 we prove some of the results of Section 4.

## 2 Fractional integrals and derivatives

An exhaustive source on fractional integrals and derivatives is the book by Samko et al. (1993). For convenience to the reader, we provide below definitions of those fractional operators that are used throughout this paper and also list a number of their properties.

Consider the interval $[0, a]$ and let $s \in [0, a]$. An integral over $[0, s]$ is called left-sided and one over $[s, a]$ is called right-sided. The right-sided fractional *integral* of order $\alpha > 0$ on an interval $[0, a]$ of a function $f \in L^1[0, a]$ is defined by

$$(I_{a-}^\alpha f)(s) = \frac{1}{\Gamma(\alpha)} \int_0^a f(u)(u-s)_+^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_s^a f(u)(u-s)^{\alpha-1} du, \; s \in (0, a) \tag{2.1}$$

(see p. 33 in Samko et al. (1993)). The right-sided fractional *derivative* of order $0 < \alpha < 1$ on an interval $[0, a]$ of a function $\phi$ is defined by

$$(\mathcal{D}_{a-}^\alpha \phi)(u) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_0^a \phi(s)(s-u)_+^{-\alpha} ds, \; u \in (0, a) \tag{2.2}$$

(see p. 35 in Samko et al. (1993)). If $0 < \alpha < 1$ and

$$\phi(s) = (I_{a-}^\alpha f)(s), \; s \in (0, a), \tag{2.3}$$

then

$$f(u) = (\mathcal{D}_{a-}^\alpha \phi)(u), \; u \in (0, a) \tag{2.4}$$

(see pp. 29-31 in Samko et al. (1993)). Hence, $\mathcal{D}^{\alpha}_{a-}$ can be viewed as an inverse of $I^{\alpha}_{a-}$. (Heuristically, performing the differentiation over $u$ in (2.2) yields (2.1) with $\mathcal{D}^{\alpha}_{a-} = I^{-\alpha}_{a-}$.) For this reason, we will often denote in the sequel the fractional derivative $\mathcal{D}^{\alpha}_{a-}$ with $\alpha \in (0,1)$ by $I^{-\alpha}_{a-}$ and also use the notation $I^{0}_{a-}$ for the identity operator, that is, $I^{0}_{a-}f = f$. One can also show that, for any $f \in L^1[0,a]$, we have

$$(\mathcal{D}^{\alpha}_{a-} I^{\alpha}_{a-} f)(s) = f(s), \ s \in (0,a) \tag{2.5}$$

(see Samko et al. (1993), p. 24).

# 3 Representation of fBm on an interval

The following proposition relates fractional Brownian motion and the fractional integral and derivative operators on an interval introduced in Section 2. It will be used in Section 4 to construct classes of integrands for fBm on an interval $[0,a]$.

**Proposition 3.1** *Let $a > 0$ and $B^{\kappa}$ be a standard fBm with parameter $\kappa \in (-1/2, 1/2)$. Then*

$$\{B^{\kappa}(t)\}_{t\in[0,a]} \stackrel{d}{=} \left\{\sigma_1(\kappa) \int_0^a s^{-\kappa} (I^{\kappa}_{a-} u^{\kappa} 1_{[0,t)}(u))(s) dB^0(s)\right\}_{t\in[0,a]}, \tag{3.1}$$

*where*

$$\sigma_1(\kappa)^2 = \frac{\Gamma(\kappa)^2 \kappa(2\kappa+1)}{\mathrm{B}(\kappa, 1-2\kappa)} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa} \tag{3.2}$$

*and $\mathrm{B}(p,q) = \int_0^1 (1-v)^{p-1} v^{q-1} dv = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, $p,q > 0$, is the beta function.*

Proof: Suppose first that $\kappa \in (0, 1/2)$. It follows from (1.1) that $d^2\Gamma^{\kappa}(u,v) = \kappa(2\kappa+1)|u - v|^{2\kappa-1} du dv$ and hence that, for any $t_1, t_2 \in \mathbb{R}$,

$$\Gamma^{\kappa}(t_1, t_2) = \kappa(2\kappa+1) \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,t_1)}(u) 1_{[0,t_2)}(u) |u-v|^{2\kappa-1} du dv. \tag{3.3}$$

By making the change of variables $s = (uv)z/(\min(u,v)z + |u-v|)$ below, one can show that, for $\kappa \in (0,1/2)$ and $u,v \in [0,a]$,

$$|u-v|^{2\kappa-1} = \frac{(uv)^\kappa}{\mathrm{B}(\kappa, 1-2\kappa)} \int_0^a s^{-2\kappa}(v-s)_+^{\kappa-1}(u-s)_+^{\kappa-1}ds. \tag{3.4}$$

Then, for $t_1, t_2 \in [0,a]$, we get from (3.3), (3.4) and definition (2.1) that

$$\Gamma^\kappa(t_1,t_2) = \frac{\Gamma(\kappa)^2\kappa(2\kappa+1)}{\mathrm{B}(\kappa,1-2\kappa)} \int_0^a \left(s^{-\kappa}(I_{a-}^\kappa u^\kappa 1_{[0,t_1)}(u))(s)\right)\left(s^{-\kappa}(I_{a-}^\kappa u^\kappa 1_{[0,t_2)}(u))(s)\right) ds.$$

Hence, the process in the right-hand side of (3.1) has the same covariance structure as a standard fBm. Since it is also Gaussian and has zero mean, it is a standard fBm.

The case $\kappa \in (-1/2, 0)$ is more delicate. Observe first that, for $\kappa \in (-1/2, 1/2)$,

$$\begin{aligned} -\Gamma(1+\kappa)s^{-\kappa}(I_{a-}^\kappa u^\kappa 1_{[0,t)}(u))(s) &= s^{-\kappa}\frac{d}{ds}\int_0^t u^\kappa (u-s)_+^\kappa du & (3.5)\\ &= \kappa s^{-\kappa}\int_s^t u^{\kappa-1}(u-s)_+^\kappa du - \left(\frac{t}{s}\right)^\kappa (t-s)_+^\kappa. & (3.6)\end{aligned}$$

Indeed, by changing, in the relation below, the order of integration and performing integration by parts, we obtain that, for any $v \in [0,a]$,

$$\begin{aligned}\kappa\int_v^t\int_s^t u^{\kappa-1}(u-s)_+^\kappa du\, ds - \int_v^t t^\kappa (t-s)_+^\kappa ds &= \kappa\int_v^t du\, u^{\kappa-1}\int_v^u (u-s)^\kappa ds - \frac{t^\kappa(t-v)^{\kappa+1}}{\kappa+1}\\ = \frac{\kappa}{\kappa+1}\int_v^t u^{\kappa-1}(u-v)^{\kappa+1}du - \frac{t^\kappa(t-v)^{\kappa+1}}{\kappa+1} &= -\int_v^t u^\kappa(u-v)_+^\kappa du. \qquad (3.7)\end{aligned}$$

The equality of (3.5) and (3.6) then follows by taking the derivative of both sides in (3.7).

The idea now is to view (3.5) or (3.6) as the definition of the left-hand side of (3.5) for $\kappa$ complex, $|\kappa| < 1/2$. Let $t_1, t_2 \in [0,a]$ be fixed and $\kappa \in \mathbb{C}$ be such that $|\kappa| < 1/2$. Then one can verify (see Appendix) that the function

$$f(\kappa) = \sigma_1(\kappa)^2\int_0^a s^{-2\kappa}(I_{a-}^\kappa u^\kappa 1_{[0,t_1)}(u))(s)(I_{a-}^\kappa u^\kappa 1_{[0,t_2)}(u))(s)ds$$

is analytic on $\{\kappa : |\kappa| < 1/2\}$. On the other hand, the function

$$g(\kappa) = \Gamma^\kappa(t_1, t_2) = \frac{1}{2}\{|t_1|^{2\kappa+1} + |t_2|^{2\kappa+1} - |t_2 - t_1|^{2\kappa+1}\}$$

is also analytic on $\{\kappa : |\kappa| < 1/2\}$. Since the analytic functions $f$ and $g$ coincide for real $\kappa \in (0, 1/2)$, standard results of complex analysis (see, for example, Conway (1995), p. 79) imply that they have to coincide on $\{\kappa : |\kappa| < 1/2\}$. In particular, $f(\kappa) = g(\kappa)$ for real $\kappa \in (-1/2, 0)$. But this means that the Gaussian zero mean process on the right-hand side of (3.1) has the same covariance structure as a standard fBm for $\kappa \in (-1/2, 0)$ as well. Hence, it is a standard fBm. $\square$

**Remarks**

1. The representation of fractional Brownian motion with the kernel function on the right-hand side of (3.5) is also given in Kleptsyna et al. (1999). The representation with the kernel function in (3.6) can be found in Norros et al. (1999). Using again an analiticity argument, one can also show that, for $\kappa \in (-1/2, 1/2)$ and $s, t \in [0, a]$,

$$\Gamma(\kappa+1)s^{-\kappa}(I^\kappa_{a-}u^\kappa 1_{[0,t)}(u))(s) = (t-s)^\kappa_+ \; {}_2F_1\left(-\kappa, \kappa, \kappa+1, 1-\frac{t}{s}\right),$$

   where ${}_2F_1$ is the so-called Gauss hypergeometric function. For more details on ${}_2F_1$, see Decreusefond and Üstünel (1999).

2. Heuristically, the equality of (3.5) and (3.6) can be obtained as follows. If $\delta_{\{t-s\}}$ denotes the delta function at a point $t-s$, we get that

$$\frac{d}{ds}\int_0^t u^\kappa(u-s)^\kappa_+ du = \frac{d}{ds}\int_0^a (v+s)^\kappa v^\kappa 1_{\{0<v<t-s\}}(v)dv = \kappa\int_0^a (v+s)^{\kappa-1}v^\kappa 1_{\{0<v<t-s\}}(v)dv$$
$$-\int_0^a (v+s)^\kappa v^\kappa \delta_{\{t-s\}}(v)dv = \kappa\int_s^t u^{\kappa-1}(u-s)^\kappa_+ du - t^\kappa(t-s)^\kappa_+.$$

## 4 Integrands for fBm on an interval

Let $\{B^\kappa(u)\}_{u\in[0,a]}$ be a standard fBm with parameter $\kappa \in (-1/2, 1/2)$. Let also $\mathcal{E}_a$ denote the set of all elementary (step) functions on an interval $[0,a]$, that is, functions of the following type

$$f(u) = \sum_{k=1}^{n} f_k 1_{[u_k,u_{k+1})}(u), \ f_k \in \mathbb{R}, \ u_k \in [0,a]. \tag{4.1}$$

For an elementary function $f \in \mathcal{E}_a$ in (4.1) define the integral with respect to fBm $B^\kappa$ in a natural way by

$$\mathcal{I}_a^\kappa(f) = \int_0^a f(u) dB^\kappa(u) = \sum_{k=1}^{n} f_k (B^\kappa(u_{k+1}) - B^\kappa(u_k)). \tag{4.2}$$

It follows by Proposition 3.1 that, for $f \in \mathcal{E}_a$,

$$\mathcal{I}_a^\kappa(f) \stackrel{d}{=} \sigma_1(\kappa) \int_0^a s^{-\kappa} (I_{a-}^\kappa u^\kappa f(u))(s) dB^0(s) \tag{4.3}$$

and hence that, for all $f, g \in \mathcal{E}_a$,

$$E(\mathcal{I}_a^\kappa(f)\mathcal{I}_a^\kappa(g)) = \sigma_1(\kappa)^2 \int_0^a s^{-2\kappa} (I_{a-}^\kappa u^\kappa f(u))(s) (I_{a-}^\kappa u^\kappa g(u))(s) ds. \tag{4.4}$$

The following theorem will be used to extend the map $\mathcal{I}_a^\kappa$ on $\mathcal{E}_a$ to functions from $\Lambda_a^\kappa$ when $\kappa \in (0, 1/2)$.

**Theorem 4.1** *For $\kappa \in (0,1/2)$, the class of functions $\Lambda_a^\kappa$, defined by* (1.2), *is a linear space with the inner product* (1.4). *Moreover, the set of elementary functions $\mathcal{E}_a$ is dense in the space $\Lambda_a^\kappa$.*

We prove Theorem 4.1 in Section 8. Let now $\kappa \in (0,1/2)$. By Theorem 4.1, since the set of elementary functions $\mathcal{E}_a$ is dense in the space $\Lambda_a^\kappa$, one can extend the map $\mathcal{I}_a^\kappa : \mathcal{E}_a \mapsto \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ to the map $\mathcal{I}_a^\kappa : \Lambda_a^\kappa \mapsto \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ in a classical way so that relation (4.4) continues to hold for functions $f, g \in \Lambda_a^\kappa$. Since the extended map $\mathcal{I}_a^\kappa$ is linear and preserves the inner products, one can say that $\Lambda_a^\kappa$ is isometric to a linear subspace of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. The question is whether this linear subspace of

$\overline{\text{sp}}_{[0,a]}(B^\kappa)$ is the space $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ itself. It is clear that $\Lambda_a^\kappa$ is isometric to $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ itself if and only if $\Lambda_a^\kappa$ is a complete inner product space. We will show in Section 5 that $\Lambda_a^\kappa$ is not a complete inner product space. Hence, it is not isometric to $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ itself. It is isometric to a proper linear subspace of $\overline{\text{sp}}_{[0,a]}(B^\kappa)$ only.

Let us show now that $|\Lambda|_a^\kappa$, given by (1.3), is a linear subspace of the class of integrands $\Lambda_a^\kappa$. Suppose that the function $f$ is such that $|f| \in \Lambda_a^\kappa$ so that $f \in \Lambda_a^\kappa$ as well. Then, as in the proof of Proposition 3.1, by the Fubini's theorem we get

$$\int_0^a s^{-2\kappa}[(I_{a-}^\kappa u^\kappa f(u))(s)]^2 ds = \frac{\mathrm{B}(\kappa, 1-2\kappa)}{\Gamma(\kappa)^2} \int_0^a \int_0^a f(u)f(v)|u-v|^{2\kappa-1} du dv, \tag{4.5}$$

which shows that $|\Lambda|_a^\kappa \subset \Lambda_a^\kappa$. A linear space $|\Lambda|_a^\kappa$ becomes an inner product space if the inner product on $|\Lambda|_a^\kappa$ is defined by $(f,g)_{|\Lambda|_a^\kappa} = (f,g)_{\Lambda_a^\kappa}$ for $f, g \in |\Lambda|_a^\kappa$. It follows by (1.4) and (4.5) that

$$(f,g)_{|\Lambda|_a^\kappa} = \frac{\sigma_1(\kappa)^2 \mathrm{B}(\kappa, 1-2\kappa)}{\Gamma(\kappa)^2} \int_0^a \int_0^a f(u)f(v)|u-v|^{2\kappa-1} du dv. \tag{4.6}$$

Observe, however, that the constant in (4.6) equals $\kappa(2\kappa+1)$ because $(1_{[0,t_1)}, 1_{[0,t_2)})_{|\Lambda|_a^\kappa} = \Gamma^\kappa(t_1,t_2) = \kappa(2\kappa+1)\int_0^a\int_0^a 1_{[0,t_1)}(u)1_{[0,t_2)}(v)|u-v|^{2\kappa-1}dudv$ for $t_1, t_2 \in [0,a]$ by relation (3.3). In other words, the inner product on $|\Lambda|_a^\kappa$ equals (1.5) as well.

**Remarks**

1. An $\mathbb{R}$–analogue of the space of integrands $|\Lambda|_a^\kappa$ is the class of functions

$$|\Lambda|^\kappa = \left\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} du dv < \infty \right\}, \tag{4.7}$$

for $\kappa \in (0, 1/2)$ (see Pipiras and Taqqu (2000)). It is also an inner product space with the inner product

$$(f,g)_{|\Lambda|^\kappa} = \kappa(2\kappa+1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1} du dv. \tag{4.8}$$

Moreover, as shown in Pipiras and Taqqu (2000), $|\Lambda|^\kappa$ is a strict subspace of the class of integrands $\Lambda^\kappa$, given by (1.6), and that the inner products (1.7) and (4.8) coincide when $f, g$ belong to the smaller class $|\Lambda|^\kappa$.

2. For $\kappa \in (0, 1/2)$, the following useful inclusions hold:

$$L^2[0,a] \subset L^{2/2\kappa+1}[0,a] \subset |\Lambda|_a^\kappa \subset \Lambda_a^\kappa. \tag{4.9}$$

The first inclusion in (4.9) is obvious since $2/2\kappa + 1 < 2$ for $\kappa > 0$. According to Proposition 4.2 in Pipiras and Taqqu (2000), we have $\|f\|_{|\Lambda|^\kappa} \leq c_\kappa \|f\|_{L^{2/2\kappa+1}(\mathbb{R})}$ for some $c_\kappa > 0$ and all $f \in L^{2/2\kappa+1}(\mathbb{R})$. This implies the second inclusion in (4.9), since

$$\|f\|_{|\Lambda|_a^\kappa} = \|f 1_{[0,a]}\|_{|\Lambda|^\kappa} \leq c_\kappa \|f 1_{[0,a]}\|_{L^{2/2\kappa+1}(\mathbb{R})} = c_\kappa \|f\|_{L^{2/2\kappa+1}[0,a]}.$$

Let us also observe that the inclusion $L^2[0,a] \subset |\Lambda|_a^\kappa$ is easy to verify directly. Indeed, by using the inequality $2|f(u)||f(v)| \leq |f(u)|^2 + |f(v)|^2$ and symmetry below, we get

$$\int_0^a \int_0^a |f(u)||f(v)||u-v|^{2\kappa-1} du dv \leq \int_0^a \int_0^a |f(u)|^2 |u-v|^{2\kappa-1} du dv \leq \frac{a^{2\kappa}}{\kappa} \int_0^a |f(u)|^2 du. \tag{4.10}$$

Suppose now that $\kappa \in (-1/2, 0)$. Set $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi(s))(u)$ for some function $\phi \in L^2[0,a]$. It is well-defined for $\phi \in L^2[0,a]$ because the function $s^\kappa \phi(s)$ is in $L^1[0,a]$ for $\kappa > -1/2$ by the Hölder's inequality. Property (2.5) then implies

$$s^{-\kappa}(I_{a-}^\kappa u^\kappa f(u))(s) = s^{-\kappa}(\mathcal{D}_{a-}^{-\kappa} u^\kappa u^{-\kappa}(I_{a-}^{-\kappa} z^\kappa \phi(z))(u))(s) = \phi(s)$$

and hence that

$$\int_0^a s^{-2\kappa}[(I_{a-}^\kappa u^\kappa f(u))(s)]^2 ds = \int_0^a \phi(s)^2 ds < \infty.$$

Based on this observation and relations (4.3) and (4.4), it is natural to introduce the class of functions $\Lambda_a^\kappa$ given by (1.8). The following result is analogous to Theorem 4.1 and deals with the case $\kappa \in (-1/2, 0)$.

**Theorem 4.2** *For $\kappa \in (-1/2,0)$, the class of functions $\Lambda_a^\kappa$, defined by (1.8), is a linear space with the inner product (1.9) or (1.10). Moreover, the set of elementary functions $\mathcal{E}_a$ is dense in the space $\Lambda_a^\kappa$.*

We prove Theorem 4.2 in Section 8. By Theorem 4.2, since the set of elementary functions $\mathcal{E}_a$ is dense in $\Lambda_a^\kappa$, one can extend the map $\mathcal{I}_a^\kappa : \mathcal{E}_a \mapsto \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ to the map $\mathcal{I}_a^\kappa : \Lambda_a^\kappa \mapsto \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ so that it is linear and preserves inner products.

The space $\Lambda_a^\kappa$ can be viewed as a class of integrands for fBm on an interval $[0,a]$. It is also isometric to a linear subspace of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. Is this linear subspace the space $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ itself? Or, equivalently, is the space of functions $\Lambda_a^\kappa$ a complete inner product space? The answer is an obvious yes. Indeed, suppose that $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $\Lambda_a^\kappa$. Then. the sequence of functions $\phi_{f_n}$, associated with the functions $f_n$ by definition (1.8), is Cauchy in $L^2[0,a]$. Since the space $L^2[0,a]$ is complete, there is a function $\phi \in L^2[0,a]$ such that $\phi_{f_n} \to \phi$ in $L^2[0,a]$. If $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi(s))(u)$, then $f_n \to f$ in $\Lambda_a^\kappa$ since $\phi_{f_n} \to \phi$ in $L^2[0,a]$. This shows the completeness of $\Lambda_a^\kappa$.

**Remarks**

1. An $\mathbb{R}$-analogue of the space of integrands $\Lambda_a^\kappa$ with $\kappa \in (-1/2,0)$ is the class of functions

$$\Lambda^\kappa = \left\{ f : \exists\ \phi_f \in L^2(\mathbb{R}) \text{ such that } f = I_-^{-\kappa}\phi_f \right\} \tag{4.11}$$

(see Pipiras and Taqqu (2000)). The class $\Lambda^\kappa$ is an inner product space with the inner product

$$(f,g)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (\mathbf{D}_-^{-\kappa} f)(s)(\mathbf{D}_-^{-\kappa} g)(s)ds \tag{4.12}$$

$$= \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}} \phi_f(s)\phi_g(s)ds, \tag{4.13}$$

where $\mathbf{D}_-^{-\kappa}$, $\kappa \in (-1/2, 0)$, is the so-called Marchaud fractional derivative of order $(-\kappa)$ which has the property $\mathbf{D}_-^{-\kappa} I_-^{-\kappa}\phi = \phi$ for $\kappa \in (-1/2, 0)$ and $\phi \in L^2(\mathbb{R})$. Observe that, when $\kappa \in (-1/2, 0)$, both the space $\Lambda^\kappa$ and the space $\Lambda_a^\kappa$ are built up by exploiting the same idea.

2. We emphasize again that the inner product space $\Lambda_a^\kappa$ is *complete*, if $\kappa \in (-1/2, 0)$, and *incomplete*, if $\kappa \in (0, 1/2)$. The proof in Section 5 below and the proof of completeness preceding these remarks show that this difference in completeness is a consequence of the following two facts:

   (a) if $\kappa \in (-1/2, 0)$, then the equation

   $$s^{-\kappa}(I_{a-}^{\kappa} u^\kappa f(u))(s) = s^{-\kappa}(\mathcal{D}_{a-}^{-\kappa} u^\kappa f(u))(s) = \phi(s) \tag{4.14}$$

   has a solution $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi(s))(u)$ for every $\phi \in L^2[0, a]$,

   (b) when $\kappa \in (0, 1/2)$, however, there are functions $\phi \in L^2[0, a]$ for which the equation

   $$s^{-\kappa}(I_{a-}^{\kappa} u^\kappa f(u))(s) = \phi(s) \tag{4.15}$$

   is not solvable. The idea here is that, since $I_{a-}^{\kappa}$ is the *integral* operator, the left-hand side of (4.15) must satisfy some *smoothness* conditions (for example, one can take its weighted fractional derivative $(\mathcal{D}_{a-}^{\kappa} s^\kappa)$) whereas such smoothness conditions need not hold for a general $\phi \in L^2[0, a]$.

# 5 Incompleteness of classes of integrands when $0 < \kappa < 1/2$

The goal of this section is to show that neither the space $|\Lambda|_a^\kappa$ nor the space $\Lambda_a^\kappa$ is a complete inner product space when $\kappa \in (0, 1/2)$. We will give a proof by first providing an equivalent criteria for the completeness and then by showing that it does not hold. We begin with a number of lemmas which will be used in the sequel. The parameter $\kappa$ is in the range $(0, 1/2)$ throughout.

**Lemma 5.1** *Let $0 \le c < b \le a$ and $\kappa \in (0,1/2)$. Then there is a function $f_{c,b}$ such that*

$$s^{-\kappa}\left(I^{\kappa}_{a-} u^{\kappa} f_{c,b}(u)\right)(s) = 1_{[c,b)}(s), \text{ for all } 0 \le s \le a. \tag{5.1}$$

PROOF: Heuristically, by solving equation (5.1), we obtain that

$$f_{c,b}(u) = u^{-\kappa}\left(I^{-\kappa}_{a-} s^{\kappa} 1_{[c,b)}(s)\right)(u) = u^{-\kappa}\left(\mathcal{D}^{\kappa}_{a-} s^{\kappa} 1_{[c,b)}(s)\right)(u). \tag{5.2}$$

To show that this function indeed satisfies relation (5.1), we may assume by linearity that $c = 0$. We may also show (5.1) for $s < b$ only, since for $s \ge b$ it is obvious (both sides are zero). As in (3.5) and (3.6) (see also remark 2 in Section 3 for heuristics), we have that

$$-\Gamma(1-\kappa)\left(\mathcal{D}^{\kappa}_{a-} s^{\kappa} 1_{[c,b)}(s)\right)(u) = \kappa \int_u^b s^{\kappa-1}(s-u)^{-\kappa} ds - b^{\kappa}(b-u)_+^{-\kappa}.$$

Then, for $s < b$,

$$\begin{aligned}
&-\Gamma(1-\kappa)\Gamma(\kappa) s^{-\kappa}\left(I^{\kappa}_{a-} u^{\kappa} u^{-\kappa}(\mathcal{D}^{\kappa}_{a-} z^{\kappa} 1_{[c,b)}(z))(u)\right)(s) \\
&= s^{-\kappa} \int_s^a \left\{ \kappa \int_u^b z^{\kappa-1}(z-u)^{-\kappa} dz - b^{\kappa}(b-u)_+^{-\kappa} \right\} (u-s)^{\kappa-1} du \\
&= s^{-\kappa}\kappa \int_s^b dz\, z^{\kappa-1} \int_s^z (z-u)^{-\kappa}(u-s)^{\kappa-1} du - s^{-\kappa} b^{\kappa} \int_s^b (b-u)_+^{-\kappa}(u-s)^{\kappa-1} du \\
&= \mathrm{B}(1-\kappa,\kappa)\left(\kappa s^{-\kappa} \int_s^b z^{\kappa-1} dz - s^{-\kappa} b^{\kappa}\right) = -\Gamma(1-\kappa)\Gamma(\kappa),
\end{aligned}$$

since $\int_s^z (z-u)^{-\kappa}(u-s)^{\kappa-1} du = \mathrm{B}(1-\kappa,\kappa) = \Gamma(1-\kappa)\Gamma(\kappa)$. This shows (5.1). $\square$

The proof of the following lemma is similar to that of Lemma 5.6 in Pipiras and Taqqu (2000). We include it for convenience to the reader.

**Lemma 5.2** *Let $\kappa \in (0,1/2)$. The inner product space $\Lambda^{\kappa}_a$ is complete if and only if, for every $\phi \in L^2[0,a]$, there is a function $f_{\phi} \in \Lambda^{\kappa}_a$ such that*

$$s^{-\kappa}\left(I^{\kappa}_{a-} u^{\kappa} f_{\phi}(u)\right)(s) = \phi(s) \text{ a.e. } ds. \tag{5.3}$$

PROOF: Suppose that the inner product space $\Lambda_a^\kappa$ is complete and let $\phi \in L^2[0,a]$. There is a sequence of elementary functions $\phi_n$ such that $\phi_n \to \phi$ in $L^2[0,a]$. By Lemma 5.1, we can express the elementary functions $\phi_n$ as $\phi_n = s^{-\kappa}(I_{a-}^\kappa u^\kappa f_n(u))(s)$, for some $f_n \in \Lambda_a^\kappa$. Since the sequence $\{\phi_n\}_{n\geq 1}$ is Cauchy in $L^2[0,a]$, the sequence $\{f_n\}_{n\geq 1}$ is Cauchy in $\Lambda_a^\kappa$. Then, the completeness of $\Lambda_a^\kappa$ implies that there is $f \in \Lambda_a^\kappa$ such that $\phi_n = s^{-\kappa}(I_{a-}^\kappa u^\kappa f_n(u))(s) \to s^{-\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)$ in $L^2[0,a]$. Since $\phi_n \to \phi$ in $L^2[0,a]$ as well, relation (5.3) holds with $f_\phi = f$.

Conversely, suppose that (5.3) holds and let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $\Lambda_a^\kappa$. Then the sequence $\phi_n(s) = s^{-\kappa}(I_{a-}^\kappa u^\kappa f_n(u))(s)$ is Cauchy in $L^2[0,a]$. Since $L^2[0,a]$ is complete, there is $\phi \in L^2[0,a]$ such that $\phi_n \to \phi$ in $L^2[0,a]$. By the assumption, there is $f_\phi \in \Lambda_a^\kappa$ such that (5.3) holds. Since $\phi_n \to \phi$ in $L^2[0,a]$ implies $f_n \to f_\phi$ in $\Lambda_a^\kappa$, the space $\Lambda_a^\kappa$ is complete. $\square$

**Lemma 5.3** *Let $\kappa \in (0,1/2)$. There are continuous functions $\psi$ on $[0,a]$ such that the equation*

$$(I_{a-}^\kappa g)(s) = \psi(s) \quad \text{a.e. } ds \quad \textit{on } [0,a] \tag{5.4}$$

*has no solution in $g \in L^1[0,a]$.*

PROOF: The proof of the lemma is by contradiction. Suppose that the equation (5.4) has a solution $g_\psi \in L^1[0,a]$ for any $\psi \in L^2[0,a]$. By (2.4),

$$g_\psi(u) = -\frac{1}{\Gamma(1-\kappa)}\frac{d}{du}\int_u^a \psi(s)(s-u)_+^{-\kappa}ds,\ u \in (0,a)$$

(see also Section 2.1 in Samko et al. (1993)). Since $g_\psi(u)$ is expressed as a derivative, the function

$$U_\psi(u) = \int_0^a \psi(s)(s-u)_+^{-\kappa}ds = \int_{\mathbb{R}} \psi(s+u)1_{\{0<s+u<a\}}s_+^{-\kappa}ds,\ u \in (0,a),$$

is differentiable almost everywhere on $(0,a)$. However, as shown in Pipiras and Taqqu (2000) (see Lemma 5.7), there are functions $\psi \in L^2[0,a]$ such that $U_\psi$ is not differentiable on a set of positive

Lebesgue measure. For example, when $a = 1$ (otherwise, use appropriate scaling), for the function $\psi$ we can take the real or imaginary part of

$$\psi^*(u) = c_0 \sum_{n=1}^{\infty} b^{-pn} e^{ib^n u}, \tag{5.5}$$

where $b > 1$, $0 < p < \kappa$ and $c_0 = (\Gamma(1-\kappa))^{-1} e^{-i\pi(1-\kappa)/2}$. Indeed, for $v \in (0,1)$, we have

$$U_{\psi^*}(v) = c_0 \int_0^{1-v} s_+^{-\kappa} \left[ \sum_{n=1}^{\infty} b^{-pn} e^{ib^n s} e^{ib^n v} \right] ds = c_0 \sum_{n=1}^{\infty} b^{-pn} \left[ \int_0^{1-v} e^{ib^n s} s_+^{-\kappa} ds \right] e^{ib^n v}$$

$$= c_0 \sum_{n=1}^{\infty} b^{-pn} \left[ \int_0^{\infty} e^{ib^n s} s_+^{-\kappa} ds \right] e^{ib^n v} - c_0 \sum_{n=1}^{\infty} b^{-pn} \left[ \int_{1-v}^{\infty} e^{ib^n s} s_+^{-\kappa} ds \right] e^{ib^n v} =: y_1(v) + y_2(v).$$

As indicated in Pipiras and Taqqu (2000), the idea then is to show that the function $y_1$ is not differentiable on $[0,1]$ and that the function $y_2$ is differentiable on $[0,1/2]$. By making a change of variables $b^n s = z$ in $y_1$, we obtain that

$$y_1(v) = \sum_{n=1}^{\infty} b^{-(p-\kappa+1)n} e^{ib^n v}.$$

Since $b^{-(p-\kappa+1)} b = b^{\kappa-p} > 1$, the function $y_1$ is a particular case of the well-known Weierstrass function whose real and imaginary parts are nowhere differentiable functions. One can show that the function $y_2$ is differentiable on $[0,1/2]$ by using standard results from real analysis (for details, see Pipiras and Taqqu (2000)). Finally, since the function (5.5) is continuous, we obtain the result. $\Box$

By combining the previous lemmas, we obtain:

**Theorem 5.1** *Let $\kappa \in (0,1/2)$. The inner product spaces $\Lambda_a^\kappa$ and $|\Lambda|_a^\kappa$ are not complete.*

PROOF: For the inner product space $\Lambda_a^\kappa$ this theorem follows from Lemma 5.2 and Lemma 5.3. Indeed, by Lemma 5.3, there is a continuous function $\psi$ such that the equation (5.4) has no solution in $g$. Then the function $\phi(s) = s^{-\kappa}\psi(s)$ is in $L^2[0,a]$, since $\kappa < 1/2$, and the equation (5.3) has no solution in $f$. Thus, by Lemma 5.2, the inner product space $\Lambda_a^\kappa$ is not complete.

We now turn to the inner product space $|\Lambda|_a^\kappa$. Since $\Lambda_a^\kappa$ is not complete and the set of elementary (step) functions $\mathcal{E}^a$ is dense in $\Lambda_a^\kappa$, there is a Cauchy sequence $\{f_n\}_{n\geq 1} \subset \mathcal{E}^a$ which does not converge in $\Lambda_a^\kappa$. Since $\mathcal{E}^a \subset |\Lambda|_a^\kappa$ and $(g,h)_{|\Lambda|_a^\kappa} = (g,h)_{\Lambda_a^\kappa}$, for $g, h \in \mathcal{E}^a$, the sequence $\{f_n\}_{n\geq 1} \subset \mathcal{E}^a$ is also Cauchy in $|\Lambda|_a^\kappa$. If $|\Lambda|_a^\kappa$ is complete, there is $f \in |\Lambda|_a^\kappa$ such that $f_n \to f$ in $|\Lambda|_a^\kappa$. But then $f_n \to f$ in $\Lambda_a^\kappa$ as well because $|\Lambda|_a^\kappa \subset \Lambda_a^\kappa$ and $(g,h)_{|\Lambda|_a^\kappa} = (g,h)_{\Lambda_a^\kappa}$, for $g, h \in |\Lambda|_a^\kappa$. Since this is a contradiction, we obtain that the inner product space $|\Lambda|_a^\kappa$ is not complete. $\square$

# 6 The case of the positive half-axis

As mentioned in Section 1, the results on incompleteness of classes of integrands are also valid when $a = \infty$ and $\kappa \in (0, 1/2)$. When $\kappa \in (0, 1/2)$, by using the scheme described in Section 4, one can construct the following two classes of integrands on $[0, \infty)$:

$$\Lambda_\infty^\kappa = \left\{ f : [0,\infty) \mapsto \mathbb{R} \text{ such that } \int_0^\infty \left[ s^{-\kappa} \left( I_-^\kappa u^\kappa f(u) \right)(s) \right]^2 ds < \infty \right\}$$

where $(I_-^\kappa \phi)(s) = (\Gamma(\kappa))^{-1} \int_{\mathbb{R}} \phi(u)(u-s)_+^{\kappa-1} du$, $s > 0$, is a fractional integral of order $\kappa$, and

$$|\Lambda|_\infty^\kappa = \left\{ f : [0,\infty) \mapsto \mathbb{R} \text{ such that } \int_0^\infty \int_0^\infty |f(u)||f(v)||u-v|^{2\kappa-1} du dv < \infty \right\}.$$

As in the case $a > 0$, the classes of integrands $\Lambda_\infty^\kappa$ and $|\Lambda|_\infty^\kappa$ are inner product spaces with the inner products

$$(f,g)_{\Lambda_\infty^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa} \int_0^\infty s^{-2\kappa} \left( I_-^\kappa u^\kappa f(u) \right)(s) \left( I_-^\kappa u^\kappa g(u) \right)(s) ds$$

and

$$(f,g)_{|\Lambda|_\infty^\kappa} = \kappa(2\kappa+1) \int_0^\infty \int_0^\infty f(u)g(v)|u-v|^{2\kappa-1} du dv,$$

respectively. We also have that $|\Lambda|_\infty^\kappa \subset \Lambda_\infty^\kappa$ and that the corresponding inner products are equal for functions $f, g$ belonging to the smaller class $|\Lambda|_\infty^\kappa$.

**Corollary 6.1** *Let $\kappa \in (0,1/2)$. The inner product spaces $\Lambda_\infty^\kappa$ and $|\Lambda|_\infty^\kappa$ are not complete.*

PROOF: We will show that the inner product space $\Lambda_\infty^\kappa$ is not complete by contradiction. Suppose that $\Lambda_\infty^\kappa$ is complete. By Theorem 5.1, the inner product space $\Lambda_a^\kappa$ is not complete. Hence, there is a Cauchy sequence $g_n$ of functions in $\Lambda_a^\kappa$ such that it does not converge. Since $(I_-^\kappa f 1_{[0,a]})(s) = (I_{a-}^\kappa f)(s)1_{[0,a]}(s)$, for any $f \in L^1[0,a]$ and $s>0$, the sequence of functions $g_n 1_{[0,a]}$ is also Cauchy in $\Lambda_\infty^\kappa$. The completeness of $\Lambda_\infty^\kappa$ then implies that there is a function $g \in \Lambda_\infty^\kappa$ such that $g_n 1_{[0,a]} \to g$ in $\Lambda_\infty^\kappa$, that is, $s^{-\kappa}(I_-^\kappa u^\kappa g_n(u)1_{[0,a]}(u))(s)$ converges to $s^{-\kappa}(I_-^\kappa u^\kappa g(u))(s)$ in the $L^2(\mathbb{R})$-sense. Since $(I_-^\kappa u^\kappa g_n(u)1_{[0,a]}(u))(s) = 0$, for $s>a$, we obtain that $(I_-^\kappa u^\kappa g(u))(s) = 0$, for $s>a$, as well. By Lemma 6.1 below, $g(u) = 0$ a.e. for $u>a$, that is, $g(u) = g(u)1_{[0,a]}(u)$ a.e. for $u>0$. Since

$$\begin{aligned}\int_0^a \left[s^{-\kappa}\left(I_{a-}^\kappa u^\kappa(g_n(u)-g(u))\right)(s)\right]^2 ds &= \int_0^\infty \left[s^{-\kappa}\left(I_-^\kappa u^\kappa(g_n(u)1_{[0,a]}(u) - g(u)1_{[0,a]})\right)(s)\right]^2 ds \\ &= \int_0^\infty \left[s^{-\kappa}\left(I_-^\kappa u^\kappa(g_n(u)1_{[0,a]}(u) - g(u)\right)(s)\right]^2 ds \longrightarrow 0,\end{aligned}$$

as $n \to \infty$, $g_n \to g$ in $\Lambda_a^\kappa$ which is a contradiction. Hence, the inner product space $\Lambda_\infty^\kappa$ is not complete. The inner product space $|\Lambda|_\infty^\kappa$ is not complete either by arguments similar to those in the proof of Theorem 5.1. $\square$

**Lemma 6.1** *Let $\kappa \in (0,1/2)$, $a>0$ be fixed and $g \in \Lambda_\infty^\kappa$. If, for $s>a$, $(I_-^\kappa u^\kappa g(u))(s) = 0$ a.e. $ds$, then, for $u>a$, $g(u) = 0$ a.e. $du$.*

PROOF: Let $\psi_{c,b}(s) = (\Gamma(1-\kappa))^{-1}((s-c)_+^{-\kappa} - (s-b)_+^{-\kappa})$, $s \in \mathbb{R}$, be a function with $a<c<b$ and observe that it vanishes for $s \le c$. Define now the operator $I_+^\kappa$ by

$$(I_+^\kappa \psi)(u) = \frac{1}{\Gamma(\kappa)}\int_{\mathbb{R}} \psi(s)(u-s)_+^{\kappa-1} ds, \quad s \in \mathbb{R}.$$

Heuristically, by changing the order of integration below, we obtain that

$$
\begin{aligned}
0 &= \int_a^\infty (I_-^\kappa u^\kappa g(u))(s)\psi_{c,b}(s)ds = \frac{1}{\Gamma(\kappa)}\int_a^\infty \left\{\int_a^\infty u^\kappa g(u)(u-s)_+^{\kappa-1}du\right\}\psi_{c,b}(s)ds \\
&= \frac{1}{\Gamma(\kappa)}\int_a^\infty u^\kappa g(u)\left\{\int_a^\infty \psi_{c,b}(s)(s-u)_-^{\kappa-1}ds\right\}du = \int_a^\infty u^\kappa g(u)(I_+^\kappa\psi_{c,b})(u)du \qquad (6.1)
\end{aligned}
$$

and

$$
0 = \int_a^\infty u^\kappa g(u)1_{(c,b]}(u)du, \qquad (6.2)
$$

since, by Lemma 6.2 below, the function $\psi_{c,b}$ has the property $I_+^\kappa\psi_{c,b} = 1_{(c,b]}$. One way to prove relation (6.2), is to justify the change of the order of integration in (6.1) and show that

$$
\int_a^\infty \int_a^\infty |u^\kappa g(u)||\psi_{c,b}(s)|(u-s)_+^{\kappa-1}duds < \infty.
$$

This can be done for a.e. $b$. As in the proof of Lemma 5.4 in Pipiras and Taqqu (2000), by using the relation $|\psi_{c,b}| = \psi_{c,b} - 2\psi_{c,b}1_{(b,\infty)}$, we obtain that

$$
\begin{aligned}
&(\Gamma(\kappa))^{-1}\int_a^\infty |\psi_{c,b}(s)|(u-s)_+^{\kappa-1}ds \\
&\le (I_+^\kappa|\psi_{c,b}|)(u) \\
&= (I_+^\kappa\psi_{c,b})(u) - 2(I_+^\kappa\psi_{c,b}1_{(b,\infty)})(u) \\
&= (I_+^\kappa\psi_{c,b})(u) - 2(I_+^\kappa\psi_{c,b}1_{(b,\infty)})(u)1_{(b,\infty)}(u) \\
&= (I_+^\kappa\psi_{c,b})(u) - 2(I_+^\kappa\psi_{c,b})(u)1_{(b,\infty)}(u) + 2(I_+^\kappa\psi_{c,b}1_{(c,b)})(u)1_{(b,\infty)}(u) \\
&= 1_{(c,b]}(u) - 21_{(c,b]}(u)1_{(b,\infty)}(u) \\
&\quad + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_c^b (s-c)_+^{-\kappa}(u-s)_+^{\kappa-1}ds\ 1_{(b,\infty)}(u) \\
&\le 1_{(c,b]}(u) + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_c^b (s-c)_+^{-\kappa}ds\ (u-b)_+^{\kappa-1} \\
&= 1_{(c,b]}(u) + c_\kappa(b-c)^{1-\kappa}\ (u-b)_+^{\kappa-1}.
\end{aligned}
$$

It is then enough to verify that

$$\int_a^\infty |u^\kappa g(u)| 1_{(c,b]}(u) du < \infty \tag{6.3}$$

and

$$\int_a^\infty |u^\kappa g(u)| (u-b)_+^{\kappa-1} du < \infty. \tag{6.4}$$

Since $g \in \Lambda_\infty^\kappa$ implies that $(I_-^\kappa u^\kappa g(u))(s)$ is well-defined a.e. for $s > a$, that is,

$$\text{for } s > a, \quad \int_a^\infty |u^\kappa g(u)| (u-s)_+^{\kappa-1} du < \infty \quad \text{a.e. } ds, \tag{6.5}$$

the equality (6.3) follows for any $a < c < b$ (we implicitly assume that integration is in the Lebesgue's sense everywhere). If $b$ is such that (6.5) holds, then we immediately get (6.4) and hence relation (6.2) is valid. However, if (6.5) does not hold for $b$, it holds for some $b_n$ such that $b_n \to b$. Then (6.2) with $b$ follows directly by letting $n \to \infty$ in

$$0 = \int_a^\infty u^\kappa g(u) 1_{(c,b_n]}(u) du.$$

Finally, by approximating the set $\{u : a < u < n\} \cap \{u : u^\kappa g(u) > 0\}$ by unions of disjoint intervals and letting $n \to \infty$, it follows from (6.2) that $u^\kappa g(u) \le 0$ a.e. $du$ for $u > a$ and, by symmetry, that $u^\kappa g(u) \ge 0$ a.e. for $u > a$. Hence, $u^\kappa g(u) = 0$ a.e. $du$ for $u > a$ or $g(u) = 0$ a.e. $du$ for $u > a$. □

**Lemma 6.2** *Let $c < b$ be real numbers and $\kappa \in (0,1)$. Then the function*

$$\psi_{c,b}(s) = (\Gamma(1-\kappa))^{-1} \left( (s-c)_+^{-\kappa} - (s-b)_+^{-\kappa} \right)$$

*satisfies the equation*

$$(I_+^\kappa \psi_{c,b})(u) = 1_{(c,b]}(u), \; u \in \mathbb{R}.$$

PROOF: By the definition of $I_+^\kappa$ (see the proof above), we have to verify that, for $u \in \mathbb{R}$,

$$J_1(u) := \int_{\mathbb{R}} \left( (s-c)_+^{-\kappa} - (s-b)_+^{-\kappa} \right) (u-s)_+^{\kappa-1} ds$$

$$= \Gamma(\kappa)\Gamma(1-\kappa) 1_{[c,b)}(u) =: J_2(u).$$

If $u \leq c$, then $J_1(u) = J_2(u) = 0$. If $u > c$, we use the following identity, valid for $t > 0$:

$$\int_0^t v^{-\kappa}(t-v)^{\kappa-1} dv = t^{-\kappa+\kappa-1+1} \int_0^1 s^{-\kappa}(1-s)^{\kappa-1} ds = B(\kappa, 1-\kappa)$$

$$= \frac{\Gamma(\kappa)\Gamma(1-\kappa)}{\Gamma(\kappa+1-\kappa)} = \Gamma(\kappa)\Gamma(1-\kappa).$$

If $c < u \leq b$, then

$$J_1(u) = \int_c^u (s-c)_+^{-\kappa}(u-s)_+^{\kappa-1} ds = \int_0^{u-c} v^{-\kappa}(u-c-v)^{\kappa-1} dv = \Gamma(\kappa)\Gamma(1-\kappa),$$

which is also $J_2(u)$. In the case when $b < u$, we show in a similar way that

$$J_1(u) = \int_c^u (s-c)_+^{-\kappa}(u-s)_+^{\kappa-1} ds - \int_b^u (s-b)_+^{-\kappa}(u-s)_+^{\kappa-1} ds = 0 = J_2(u).$$

□

**Remark.** The result of Lemma 6.2 should not be surprising because of the relation $\psi_{c,b} = I_+^{-\kappa} 1_{(c,b]}$, where the $\kappa$-fractional derivative operator $I_+^{-\kappa}$, defined as

$$(I_+^{-\kappa} f)(s) = \frac{1}{\Gamma(1-\kappa)} \frac{d}{ds} \int_{-\infty}^s f(u)(s-u)_+^{-\kappa} du, \quad s \in \mathbb{R},$$

is the inverse of $I_+^\kappa$ (there exist other definitions for an inverse of $I_+^\kappa$ - see, for example, Section 5 in Samko et al. (1993)).

# 7 Prediction of fBm

The prediction problem for fBm is to find an explicit expression for the conditional expectation $X = E(B^\kappa(t)|B^\kappa(s), s \in [0,a])$ with some fixed $0 < a < t$. As already indicated in the introduction, we have $X \in \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ and therefore we expect $X = \int_0^a f(u)dB^\kappa(u)$ for some function $f$. When $\kappa \in (-1/2, 0)$, since the class of integrands $\Lambda_a^\kappa$ in (1.8) is complete, we know that the function $f$ exists and belongs to $\Lambda_a^\kappa$. When $\kappa \in (0, 1/2)$, however, since the class $\Lambda_a^\kappa$ in (1.2) is not complete, we may only hope that there is such a function $f$ which belongs to $\Lambda_a^\kappa$.

In this section, we will show that the above function $f$ exists not only for $\kappa \in (-1/2, 0)$ but for $\kappa \in (0, 1/2)$ as well. We will also determine this function explicitely by first providing its heuristic derivation which uses fractional integration ideas and then verifying rigorously that the obtained function is indeed the right one. When $\kappa \in (0, 1/2)$, the prediction formula was obtained by Gripenberg and Norros (1996). Our approach is different from that of Gripenberg and Norros (1996), where fractional integration is not used, and it also covers the case $\kappa \in (-1/2, 0)$.

The prediction formula for fBm can be *heuristically* derived as follows. We shall repeatedly use the (heuristic) fractional integration "rule" $I^\alpha I^\beta = I^{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{R}$, where $I^\alpha$ is some fractional integral or derivative operator. For $s \in [0, t]$, set

$$B^0(s) = \int_0^t u^{-\kappa}(I_{t-}^{-\kappa} z^\kappa 1_{[0,s)}(z))(u)dB^\kappa(u). \tag{7.1}$$

Since $v^{-\kappa}(I_{t-}^\kappa u^\kappa(u^{-\kappa} I_{t-}^{-\kappa} z^\kappa 1_{[0,s)}(z)))(v) = 1_{[0,s)}(v)$, $B^0$ is the usual Brownian motion. By using the fractional integration by parts formula $\int_0^t \phi(s)(I_{t-}^\alpha \psi)(s)ds = \int_0^t (I_{0+}^\alpha \phi)(s)\psi(s)ds$ with $(I_{0+}^\alpha \phi)(s) = \Gamma(\alpha)^{-1}\int_a^b \phi(u)(s-u)_+^{\alpha-1}du$, we can write (7.1) as $B^0(s) = \int_0^t 1_{[0,s)}(z)z^\kappa(I_{0+}^{-\kappa} u^{-\kappa}\dot{B}^\kappa(u))(z)dz$, where

$\dot{B}^\kappa$ denotes the "derivative" of $B^\kappa$, or

$$\dot{B}^0(s) = s^\kappa (I_{0+}^{-\kappa} u^{-\kappa} \dot{B}^\kappa(u))(s).$$

By applying the operator $u^\kappa I_{0+}^\kappa s^{-\kappa}$ to both sides of the last expression, integrating over $[0,s]$ and applying the fractional integration by parts formula again, we get

$$B^\kappa(s) = \int_0^t u^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[0,s)}(z))(u)dB^0(u). \tag{7.2}$$

Since $B^0$ and $B^\kappa$ can be defined through each other as in (7.1) and (7.2), we get $\sigma\{B^\kappa(s), s \in [0,a]\} = \sigma\{B^0(s), s \in [0,a]\}$ up to sets of measure zero, where $\sigma\{B^\kappa(s), s \in [0,a]\}$ is the $\sigma$-algebra generated by the random variables $B^\kappa(s)$, $s \in [0,a]$. Then, since $B^0$ is the usual Brownian motion,

$$\begin{aligned} E(B^\kappa(t)|B^\kappa(s), s \in [0,a]) &= E\left(\int_0^t u^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[0,t)}(z))(u)dB^0(u) \Big| B^0(s), s \in [0,a]\right) \\ &= \int_0^a u^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[0,t)}(z))(u)dB^0(u). \end{aligned}$$

By writing $1_{[0,t)} = 1_{[0,a)} + 1_{[a,t)}$ and then using (7.2), we get

$$E(B^\kappa(t)|B^\kappa(s), s \in [0,a]) = B^\kappa(a) + \int_0^a u^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z))(u)dB^0(u).$$

We want to rewrite this last integral as an integral with respect to fBm. Since heuristically, $u^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z))(u) = u^{-\kappa} I_{a-}^\kappa (v^\kappa v^{-\kappa} I_{a-}^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z))(v))(u)$ and since for an integrable function $f$, $\int_0^a f(u)dB^\kappa(u) = \int_0^a u^{-\kappa}(I_{a-}^\kappa z^\kappa f(z))(u)dB^0(u)$, we expect that

$$E(B^\kappa(t)|B^\kappa(s), s \in [0,a]) = B^\kappa(a) + \int_0^a u^{-\kappa}\left(I_{a-}^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z))\right)(u)dB^\kappa(u).$$

These computations suggest the following prediction formula for fractional Brownian motion.

**Theorem 7.1** *Let $0 < a < t$, $\kappa \in (-1/2, 1/2)$ and $B^\kappa$ be a fractional Brownian motion. Then*

$$E(B^\kappa(t)|B^\kappa(s), s \in [0,a]) = B^\kappa(a) + \int_0^a \Psi_t(a,u) dB^\kappa(u), \tag{7.3}$$

*where, for $u \in (0,a)$,*

$$\Psi_t(a,u) = u^{-\kappa} \left( I_{a-}^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z)) \right)(u) \tag{7.4}$$

$$= \frac{\sin \pi\kappa}{\pi} u^{-\kappa}(a-u)^{-\kappa} \int_a^t \frac{z^\kappa (z-a)^\kappa}{z-u} dz. \tag{7.5}$$

Proof: Let us first verify that the right-hand sides of (7.4) and (7.5) are equal when $\kappa \in (0, 1/2)$. Observe that, by (2.1), for $v \in (0,a)$,

$$(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z))(v) = \frac{1}{\Gamma(\kappa)} \int_a^t z^\kappa (z-v)^{\kappa-1} dz \tag{7.6}$$

and hence, by (2.2),

$$\begin{aligned} \left( I_{a-}^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z)) \right)(u) &= -\frac{1}{\Gamma(1-\kappa)\Gamma(\kappa)} \frac{d}{du} \int_u^a \int_a^t z^\kappa (z-v)^{\kappa-1} dz \ (v-u)^{-\kappa} dv \qquad (7.7) \\ &= -\frac{\sin \pi\kappa}{\pi} \frac{d}{du} \int_a^t dz \ z^\kappa \int_u^a dv \ (z-v)^{\kappa-1}(v-u)^{-\kappa}, \end{aligned}$$

since $\Gamma(1-\kappa)\Gamma(\kappa) = \pi / \sin \pi\kappa$. By making the change of variables $v = u + (z-u)s$ and then taking $d/du$ inside the integral, we get

$$\begin{aligned} \left( I_{a-}^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z)) \right)(u) &= -\frac{\sin \pi\kappa}{\pi} \frac{d}{du} \int_a^t dz \ z^\kappa \int_0^{\frac{a-u}{z-u}} ds \ (1-s)^{\kappa-1} s^{-\kappa} \\ &= \frac{\sin \pi\kappa}{\pi} (a-u)^{-\kappa} \int_a^t \frac{z^\kappa (z-a)^\kappa}{z-u} dz. \qquad (7.8) \end{aligned}$$

When $\kappa \in (-1/2, 0)$, the right-hand sides of (7.4) and (7.5) are still equal. To see this, observe first that (7.6) with $\Gamma(\kappa) = \Gamma(1+\kappa)/\kappa$ still holds for $v \in (0,a)$, since $z \neq v$ in the range of integration. Then, (7.7) becomes (see (2.1))

$$\left( I_{a-}^{-\kappa}(I_{t-}^\kappa z^\kappa 1_{[a,t)}(z)) \right)(u) = \frac{\kappa}{\Gamma(1+\kappa)\Gamma(-\kappa)} \int_u^a \int_a^t z^\kappa (z-v)^{\kappa-1} dz \ (v-u)^{-\kappa-1} dv$$

$$= -\frac{\kappa \sin \pi\kappa}{\pi} \int_a^t dz\ z^\kappa \int_u^a dv\ (z-v)^{\kappa-1}(v-u)^{-\kappa-1}.$$

It is now easy to see that

$$\kappa \int_u^a (z-v)^{\kappa-1}(v-u)^{-\kappa-1} dv = \frac{d}{du}\int_u^a (z-v)^{\kappa-1}(v-u)^{-\kappa} dv = -\frac{(a-u)^{-\kappa}(z-a)^\kappa}{z-u},$$

where the derivative can be computed as in the case $\kappa \in (0, 1/2)$. This yields the relation (7.8) in the case $\kappa \in (-1/2, 0)$ as well.

To verify that the integral in (7.3) is well-defined for $\kappa \in (-1/2, 0)$, it is enough to check, by (1.8), that the function $s^{-\kappa}(I^\kappa_{t-} z^\kappa 1_{[a,t)}(z))(s)$ belongs to $L^2[0, a]$. The same condition needs to be verified in the case $\kappa \in (0, 1/2)$ by (1.2) and (2.5). In both cases this can be deduced from (7.6). Finally, to prove (7.3), it is enough to verify that, for all $s \in [0, a]$,

$$EB^\kappa(s)(B^\kappa(t) - B^\kappa(a)) = EB^\kappa(s) \int_0^a \Psi_t(a, u) dB^\kappa(u).$$

This follows from (1.4) and (1.9) by using the expression (7.4) for $\Psi_t(a, u)$. $\square$

# 8 The proofs of Theorem 4.1 and Theorem 4.2

We prove Theorem 4.1 and Theorem 4.2 together. Let $\kappa \in (-1/2, 1/2)$. To show that the maps (1.4) and (1.9) (or (1.10)) define inner products on linear spaces $\Lambda^\kappa_a$, given by (1.2) and (1.8), respectively, we check the least obvious condition. If $(f, f)_{\Lambda^\kappa_a} = 0$ and $\kappa \in (0, 1/2)$, then $(I^\kappa_{a-} u^\kappa f(u))(s) = 0$ a.e. $s \in [0, a]$. It follows by Lemma 2.5 in Samko et al. (1993), p. 40, that $u^\kappa f(u) = 0$ a.e. $u \in [0, a]$ and hence that $f(u) = 0$ a.e. $u \in [0, a]$. If $(f, f)_{\Lambda^\kappa_a} = 0$ and $\kappa \in (-1/2, 0)$, then $\phi_f(s) = 0$ a.e. $s \in [0, a]$, where $\phi_f \in L^2[0, a]$ is associated with the function $f$ by definition (1.8). It then follows that $f(u) = 0$ a.e. $u \in [0, a]$ as well.

Let us show that the set of elementary functions $\mathcal{E}_a$ is dense in $\Lambda_a^\kappa$ when $\kappa \in (-1/2, 1/2)$. Assume without loss of generality that $a > 1$. Since any function $\phi \in L^2[0,a]$ can be approximated in $L^2[0,a]$ by functions $s^{-\kappa}\sum_{k=1}^n b_k 1_{[c_k,d_k)}(s) = s^{-\kappa}\sum_{k=1}^n b_k(1_{[0,d_k)}(s) - 1_{[0,c_k)}(s))$ with $b_k \in \mathbb{R}$ and $0 < c_k < d_k < a$, $k = 1,\dots,n$, it is enough to show, for example, that there is a sequence of elementary functions $f_n \in \mathcal{E}_a \subset \Lambda_a^\kappa$ such that

$$h_n(\kappa) := \int_0^a s^{-2\kappa} \left| 1_{[0,1)}(s) - (I_{a-}^\kappa u^\kappa f_n(u))(s) \right|^2 ds \to 0, \quad n \to \infty. \tag{8.1}$$

Since $1_{[0,1)}(s) = (I_{a-}^\kappa u^\kappa f(u))(s)$ with $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} 1_{[0,1)})(u) = (\Gamma(1-\kappa))^{-1} u^{-\kappa}(1-u)_+^{-\kappa}$, the convergence (8.1) for some elementary functions $f_n$ follows from Lemma 8.1 below.

**Lemma 8.1** *Let $\kappa \in (-1/2, 1/2)$, $\{B^\kappa(u)\}_{u \in [0,1]}$ be a standard fBm and $f(u) = u^{-\kappa}(I_{1-}^{-\kappa} 1_{[0,1)})(u) = (\Gamma(1-\kappa))^{-1} u^{-\kappa}(1-u)_+^{-\kappa}$ for $u \in [0,1]$. Then the integral $\int_0^1 f(u)dB^\kappa(u)$ is the limit in the $L^2(\Omega)$–sense of the integrals*

$$\int_0^1 f_n(u)dB^\kappa(u) = \frac{1}{\Gamma(1-\kappa)} \sum_{l=2}^{n-2} \left(\frac{l}{n}\right)^{-\kappa} \left(1 - \frac{l}{n}\right)^{-\kappa} \left( B^\kappa\left(\frac{l+1}{n}\right) - B^\kappa\left(\frac{l}{n}\right)\right),$$

*where $f_n(u) = (\Gamma(1-\kappa))^{-1}\sum_{l=2}^{n-2}(\frac{l}{n})^{-\kappa}(1-\frac{l}{n})^{-\kappa} 1_{[\frac{l}{n},\frac{l+1}{n})}(u) = (\Gamma(1-\kappa))^{-1}\sum_{l=2}^{n-2} f(\frac{l}{n}) 1_{[\frac{l}{n},\frac{l+1}{n})}(u)$ for $u \in [0,1]$.*

**Remark.** The integral $\int_0^1 f(u)dB^\kappa(u)$ in Lemma 8.1 is of particular interest because, if $f_t(u) = u^{-\kappa}(I_{t-}^{-\kappa} 1_{[0,t)})(u) = (\Gamma(1-\kappa))^{-1} u^{-\kappa}(t-u)_+^{-\kappa}$, $t, u \in [0,a]$, then by (4.3) (and Theorems 4.1 and 4.2)

$$\int_0^t f_t(u)dB^\kappa(u) \stackrel{d}{=} \sigma_1(\kappa) \int_0^t s^{-2\kappa} B^0(s),\ t \in [0,a].$$

The process on the left-hand side is known as the fundamental martingale (see Norros et al. (1999)).

PROOF: It is enough to prove the convergence (8.1) with $a=1$. If $\kappa \in (0,1/2)$, by using relation (4.5) and estimate (4.10), we get that

$$h_n(\kappa) \le \int_0^1 s^{-2\kappa}[(I_{1-}^{\kappa} u^{\kappa}|f(u)-f_n(u)|)(s)]^2 ds \le c(\kappa)\ \|f-f_n\|^2_{L^2[0,1]} \to 0,$$

as $n \to \infty$.

The case $\kappa \in (-1/2,0)$ is more delicate. We will show that $h_n(\kappa) \to 0$ as well, by applying the dominated converegnce theorem. Let $l_n = l/n$ for $l,n \in \mathbb{N}$. As in (3.5) and (3.6), we get

$$\Gamma(1-\kappa)\Gamma(1+\kappa)(I_{1-}^{\kappa} u^{\kappa} f_n(u))(s) = \Gamma(1+\kappa)\sum_{l=2}^{n-2} l_n^{-\kappa}(1-l_n)^{-\kappa}(I_{1-}^{\kappa} u^{\kappa} 1_{[l_n,(l+1)_n)}(u))(s)$$

$$= -\kappa \int_s^1 \sum_{l=2}^{n-2} l_n^{-\kappa}(1-l_n)^{-\kappa} 1_{[l_n,(l+1)_n)}(u)\ u^{\kappa-1}(u-s)_+^{\kappa} du$$

$$+\sum_{l=2}^{n-2} l_n^{-\kappa}(1-l_n)^{-\kappa}\left((l+1)_n^{\kappa}((l+1)_n - s)_+^{\kappa} - l_n^{\kappa}(l_n-s)_+^{\kappa}\right) =: g_n^1(s) + g_n(s).$$

The summation by parts formula $\sum_{l=2}^{n-2} a_l(b_{l+1}-b_l) = -\sum_{l=2}^{n-2} b_l(a_l - a_{l-1}) + a_{n-2}b_{n-1} - a_1 b_2$ implies that

$$g_n(s) = -\sum_{l=2}^{n-2} l_n^{\kappa}(l_n-s)_+^{\kappa}\left(l_n^{-\kappa}(1-l_n)^{-\kappa} - (l-1)_n^{-\kappa}(1-(l-1)_n)^{-\kappa}\right)$$

$$+(n-2)_n^{-\kappa}(1-(n-2)_n)^{-\kappa}(n-1)_n^{\kappa}((n-1)_n - s)_+^{\kappa} - 1_n^{-\kappa}(1-1_n)^{-\kappa} 2_n^{\kappa}(2_n-s)_+^{\kappa}$$

$$= -\sum_{l=2}^{n-2} l_n^{\kappa}(l_n-s)_+^{\kappa}\left(l_n^{-\kappa} - (l-1)_n^{-\kappa}\right)(1-l_n)^{-\kappa} - \sum_{l=2}^{n-2} l_n^{\kappa}(l_n-s)_+^{\kappa}(l-1)_n^{-\kappa}\left((1-l_n)^{-\kappa} - (1-(l-1)_n)^{-\kappa}\right)$$

$$+(n-2)_n^{-\kappa}(1-(n-2)_n)^{-\kappa}(n-1)_n^{\kappa}((n-1)_n - s)_+^{\kappa} - 1_n^{-\kappa}(1-1_n)^{-\kappa} 2_n^{\kappa}(2_n-s)_+^{\kappa}$$

$$=: g_n^2(s) + g_n^3(s) + g_n^4(s) + g_n^5(s).$$

(Recall that $l_n$ stands for $l/n$.) In order to establish (8.1), we will show that, for $\kappa \in (-1/2,0)$,

$$\int_0^1 s^{-2\kappa}|g_n^5(s)|^2 ds \to 0, \quad n \to \infty, \tag{8.2}$$

$$\Gamma(1-\kappa)\Gamma(1+\kappa)(I^{\kappa}_{1-}u^{\kappa}f_n(u))(s) - g^5_n(s) = g^1_n(s) + g^2_n(s) + g^3_n(s) + g^4_n(s)$$

$$\to \Gamma(1-\kappa)\Gamma(1+\kappa)(I^{\kappa}_{1-}u^{\kappa}f(u))(s) = \Gamma(1-\kappa)\Gamma(1+\kappa)1_{[0,1)}(s) \quad \text{a.e. } s\in[0,1], \tag{8.3}$$

and

$$\sup_n |g^j_n(s)| \le g^j(s), \quad 1\le j\le 4, \tag{8.4}$$

where

$$\int_0^1 s^{-2\kappa}|g^j(s)|^2 ds < \infty, \quad 1\le j\le 4. \tag{8.5}$$

Then, by the dominated convergence theorem we will have the required convergence $h_n(\kappa)\to 0$ for $\kappa\in(-1/2,0)$ as $n\to\infty$.

The convergence (8.2) follows, since

$$\int_0^1 s^{-2\kappa}|g^5_n(s)|^2 ds \le c\int_0^{2/n} s^{-2\kappa}\left(\frac{2}{n}-s\right)^{2\kappa} ds = c\frac{2}{n}\int_0^1 z^{-2\kappa}(1-z)^{2\kappa}dz \to 0,$$

as $n\to\infty$. Let us show (8.4) and (8.5) with $j=1$. We have

$$\begin{aligned} |g^1_n(s)| &\le c_1\int_s^1 \sum_{l=2}^{n-2} l_n^{-\kappa}(1-(l+1)_n)^{-\kappa}1_{[l_n,(l+1)_n)}(u)\ u^{\kappa-1}(u-s)_+^{\kappa}du \\ &\le c_1\int_s^1 u^{-\kappa}(1-u)^{-\kappa}u^{\kappa-1}(u-s)_+^{\kappa}du = c_1\int_s^1 u^{-1}(1-u)^{-\kappa}(u-s)_+^{\kappa}du =: g^1(s), \end{aligned}$$

where the constant $c_1$ depends on $\kappa$ only. It is enough to check the convergence of the integral in (8.5) around $s=0$ only. Since $\kappa<0$, observe that, as $s$ is close to 0, the function $g^1(s)$ is bounded (up to a constant) by

$$\int_s^1 u^{-1}(u-s)^{\kappa}du = s^{\kappa}\int_1^{1/s} v^{-1}(v-1)^{\kappa}dv \le cs^{\kappa}\int_1^{\infty} v^{\kappa-1}dv \le \frac{c}{|\kappa|}s^{\kappa}.$$

It then follows that the integral in (8.5) with $j=1$ converges around $s=0$. For functions $g^2_n$, by using the inequality $|l_n^{-\kappa}-(l-1)_n^{-\kappa}| \le |\kappa|(l-1)_n^{-\kappa-1}/n$, we have

$$|g^2_n(s)| \le |\kappa|\sum_{l=2}^{n-2} l_n^{\kappa}(l_n-s)_+^{\kappa}(l-1)_n^{-\kappa-1}(1-l_n)^{-\kappa}\frac{1}{n}$$

$$\leq \quad c_2 \sum_{l=2}^{n-2} (l+1)_n^{-1} (l_n - s)_+^{\kappa} (1-(l+1)_n)^{-\kappa} \frac{1}{n}$$
$$\leq \quad c_2 \int_0^1 u^{-1} (u-s)_+^{\kappa} (1-u)^{-\kappa} du =: g^2(s),$$

where the constant $c_2$ depends on $\kappa$ only. Relation (8.5) with $j=2$ follows as in the case $j=1$ since the functions $g_1$ and $g_2$ are equal up to a constant. As for the functions $g_n^3$, one can similarly show that

$$|g_n^3(s)| \leq c_3 \int_0^1 (u-s)_+^{\kappa} (1-u)^{-\kappa-1} du = c_3 \int_0^1 z^{\kappa} (1-z)^{-\kappa-1} du = c_3 \mathrm{B}(1+\kappa, -\kappa) =: g^3(s)$$

and

$$\int_0^1 s^{-2\kappa} |g^3(s)|^2 ds = c_3^2 \mathrm{B}(1+\kappa, -\kappa)^2 \int_0^1 s^{-2\kappa} ds < \infty.$$

The proof of (8.4) and (8.5) with $j=4$ is obvious. Finally, to show the convergence (8.3), observe that

$$\lim_{n\to\infty} (\Gamma(1-\kappa)\Gamma(1+\kappa)(I_{1-}^{\kappa} u^{\kappa} f_n(u))(s) - g_n^5(s)) = \lim_{n\to\infty} (g_n^1(s) + g_n^2(s) + g_n^3(s) + g_n^4(s))$$
$$= -\kappa \int_0^1 u^{-\kappa} (u-s)_+^{\kappa} u^{\kappa-1} (1-u)^{-\kappa} du + \kappa \int_0^1 u^{\kappa} (u-s)_+^{\kappa} u^{-\kappa-1} (1-u)^{-\kappa} du$$
$$-\kappa \int_0^1 u^{\kappa} (u-s)_+^{\kappa} u^{-\kappa} (1-u)^{-\kappa-1} du + 0 = -\kappa \int_0^1 (u-s)_+^{\kappa} (1-u)^{-\kappa-1} du$$
$$= -\kappa \mathrm{B}(\kappa+1, -\kappa) 1_{[0,1)}(s) = \Gamma(1-\kappa)\Gamma(1+\kappa) 1_{[0,1)}(s),$$

a.e. $s \in [0,1]$. □

# A Appendix

In the proof of Proposition 3.1, we used the fact that the function

$$f(\kappa) = \sigma_1(\kappa)^2 \int_0^a s^{-2\kappa} (I_{a-}^{\kappa} u^{\kappa} 1_{[0,t_1)}(u))(s) (I_{a-}^{\kappa} u^{\kappa} 1_{[0,t_2)}(u))(s) ds,$$

where $t_1, t_2 \in [0,a]$, is analytic for complex $|\kappa| < 1/2$. The analiticity of $f$ can be verified directly as follows. By (3.6), the function $f$ equals

$$f(\kappa) = (\Gamma(1+\kappa))^{-2}\sigma_1(\kappa)^2 \int_0^a s^{-2\kappa} \left\{ \kappa \int_s^{t_1} u^{\kappa-1}(u-s)_+^\kappa du - t_1^\kappa (t_1-s)_+^\kappa \right\}$$
$$\left\{ \kappa \int_s^{t_2} u^{\kappa-1}(u-s)_+^\kappa du - t_2^\kappa (t_2-s)_+^\kappa \right\} ds,$$

where $\sigma_1(\kappa)^2$ is defined by (3.2). Since $(\Gamma(1+\kappa))^{-2}\sigma_1(\kappa)^2$ is analytic on $|\kappa| < 1/2$, it is enough to show that the function

$$\int_0^a s^{-2\kappa} \left\{ \kappa \int_s^{t_1} u^{\kappa-1}(u-s)_+^\kappa du - t_1^\kappa (t_1-s)_+^\kappa \right\} \left\{ \kappa \int_s^{t_2} u^{\kappa-1}(u-s)_+^\kappa du - t_2^\kappa (t_2-s)_+^\kappa \right\} ds$$
$$= \kappa^2 \int_0^a s^{-2\kappa} \int_s^{t_1} u^{\kappa-1}(u-s)_+^\kappa du \int_s^{t_2} u^{\kappa-1}(u-s)_+^\kappa du\, ds - \kappa \int_0^a s^{-2\kappa} \int_s^{t_1} u^{\kappa-1}(u-s)_+^\kappa du\, t_2^\kappa (t_2-s)_+^\kappa ds$$
$$-\kappa \int_0^a s^{-2\kappa} \int_s^{t_2} u^{\kappa-1}(u-s)_+^\kappa du\, t_1^\kappa (t_1-s)_+^\kappa ds + \kappa \int_0^a s^{-2\kappa} t_1^\kappa (t_1-s)_+^\kappa t_2^\kappa (t_2-s)_+^\kappa ds$$
$$=: \kappa^2 f_1(\kappa) - \kappa f_2(\kappa) - \kappa f_3(\kappa) + f_4(\kappa)$$

is analytic on $|\kappa| < 1/2$, or, equivalently, that the functions $f_1(\kappa)$, $f_2(\kappa)$, $f_3(\kappa)$ and $f_4(\kappa)$ are analytic on $|\kappa| < 1/2$. Assume without loss of generality that $t_1 < t_2$ and consider the function $f_4(\kappa) = \int_0^{t_1} e^{\kappa \ln s^{-2} t_1(t_1-s)t_2(t_2-s)} ds$ first. For $h \in \mathbb{C}$, $h \neq 0$, we have that $\frac{1}{h}(f_4(\kappa+h) - f_4(\kappa)) = \int_0^{t_1} f_4(\kappa,h,s)ds$, where

$$f_4(\kappa,h,s) = e^{\kappa \ln s^{-2} t_1(t_1-s)t_2(t_2-s)} \, \frac{e^{h \ln s^{-2} t_1(t_1-s)t_2(t_2-s)} - 1}{h}.$$

Observe now that, for $u > 0$ and $h = h_1 + ih_2$,

$$\frac{e^{ih \ln u} - 1}{h} = \frac{e^{h_1 \ln u} e^{ih_2 \ln u} - 1}{h_1 + ih_2} = e^{ih_2 \ln u} \frac{e^{h_1 \ln u} - 1}{h_1 + ih_2} + \frac{e^{ih_2 \ln u} - 1}{h_1 + ih_2}$$

and hence that $|h|^{-1}|e^{h \ln u} - 1| \leq 2|\ln u| e^{|h_1 \ln u|} \leq 2|\ln u| e^{\epsilon |\ln u|}$, if $|h| < \epsilon$. This implies that

$$|h|^{-1}|e^{h \ln s^{-2} t_1(t_1-s)t_2(t_2-s)} - 1| \leq 2|\ln s^{-2} t_1(t_1-s)t_2(t_2-s)| \; e^{\epsilon |\ln s^{-2} t_1(t_1-s)t_2(t_2-s)|},$$

for $|h| < \epsilon$. Choose $\epsilon > 0$ so that $\int_0^1 s^{-2\kappa} s^{-2\epsilon} ds < \infty$. Since

$$\int_0^{t_1} s^{-2\kappa_1} t_1^{\kappa_1}(t_1 - s)_+^{\kappa_1} t_2^{\kappa_1}(t_2 - s)_+^{\kappa_1} |\ln s^{-2} t_1(t_1 - s)t_2(t_2 - s)|\; e^{\epsilon|\ln s^{-2} t_1(t_1-s)t_2(t_2-s)|} ds < \infty,$$

where $\kappa_1 = \Re\kappa$ of $|\kappa| < 1/2$, we obtain from the dominated convergence theorem that

$$\frac{d}{d\kappa} f_4(\kappa) = \lim_{h\to 0} \frac{f_4(\kappa + h) - f_4(\kappa)}{h} = \int_0^{t_1} s^{-2\kappa} t_1^{\kappa}(t_1 - s)_+^{\kappa} t_2^{\kappa}(t_2 - s)_+^{\kappa} \ln s^{-2} t_1(t_1 - s)t_2(t_2 - s) ds$$

ande hence that the function $f_4$ is analytic on $|\kappa| < 1/2$.

For the function $f_3(\kappa) = \int_0^{t_1} e^{\kappa \ln s^{-2} t_1(t_1-s)} \int_s^{t_2} u^{-1} e^{\kappa \ln u(u-s)} du\; ds$, write

$$\begin{aligned}\frac{f_3(\kappa + h) - f_3(\kappa)}{h} &= \int_0^{t_1} e^{\kappa \ln s^{-2} t_1(t_1-s)} \frac{e^{h \ln s^{-2} t_1(t_1-s)} - 1}{h} \int_s^{t_2} u^{-1} e^{(\kappa+h)\ln u(u-s)} du\; ds \\ &+ \int_0^{t_1} e^{\kappa \ln s^{-2} t_1(t_1-s)} \int_s^{t_2} u^{-1} e^{\kappa \ln u(u-s)} \frac{e^{h \ln u(u-s)} - 1}{h} du\; ds\end{aligned}$$

and use the arguments above to conclude that

$$\begin{aligned}\frac{d}{d\kappa} f_3(\kappa) &= \int_0^{t_1} s^{-2\kappa} t_1^{\kappa}(t_1 - s)^{\kappa} \ln s^{-2} t_1(t_1 - s) \int_s^{t_2} u^{\kappa-1}(u - s)^{\kappa} du\; ds \\ &+ \int_0^{t_1} s^{-2\kappa} t_1^{\kappa}(t_1 - s)^{\kappa} \int_s^{t_2} u^{\kappa-1}(u - s)^{\kappa} \ln u(u - s) du\; ds\end{aligned}$$

and hence that the function $f_3$ is analytic on $|\kappa| < 1/2$. The proof that the functions $f_2$ and $f_1$ are analytic on $|\kappa| < 1/2$ is similar.

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