CHAPTER 1

The Real Number System

(This is chapter 1 of *A New Course of Analysis: The Main Ideas*, by Gabriel Stolzenberg. Updated January, 2006.)

Analysis rests upon and lives off the real number system, the core features of which are that it contains the rationals as a dense subset but, unlike the rationals, is complete. Here we present one unhurried development of it that readers are unlikely to have seen. In it, a real number is a set of rational intervals, any two of which intersect and among which are ones of arbitrarily small length. Two such sets are said to be equal as real numbers if their union is again one. Some mathematicians object to calling anything *the* real number system on the grounds that there is more than one candidate—Dedekind cuts, Cauchy sequences of rationals, others—and no apparent reason to privilege a particular one. I agree. But the lesson that I draw from this is that we should replace 'the' by 'a,' where a real number system is any set, together with certain relations (order and equality) and operations (arithmetic), that is isomorphic to the one that we use here. Although it may not look like it, this is an axiomatic definition, albeit very different from the usual one. It also is the custom to consider, not objects with an equality relation as we do here, but equivalence classes of such objects. This is thought to make things simpler (except, perhaps, for equivalence classes of functions that are defined almost everywhere with respect to some measure, with respect to the relation of equality almost everywhere.) But the belief that this makes things simpler is a comparitive assessment of two ways of doing mathematics and few, if any, of those who confidently make this judgment have practiced mathematics both ways.

What seems most distinctive about our approach is that, instead of defining the real number system directly in terms of the rationals (or introducing a set of axioms tantamount to such a definition), we focus first on a largely ignored intermediary system, the set of all intervals with rational endpoints. In the first two sections, this set is endowed with an order relation, a wannabe equivalence relation that is not transitive, and counterparts to the usual arithmetic operations, all of which extend those on the rationals if we consider the latter as intervals of length zero. One possible reason that this intermediary system has been ignored is that it lacks basic properties that are possessed both by the rationals and the reals, e.g., as noted above, transitivity of equality. And yet, in the third section, when we establish these properties for the real number system, we do so by reasoning about this intermediate system of intervals, for which they do not hold. Moreover, in the course of this reasoning, we learn what it is about real numbers that makes these proofs work for them qua sets of rational intervals but not for the intervals themselves nor for more general sets of them.

Although the treatment of max and min is not original, it, too, is likely to be unfamiliar. Indeed, the fact that neither the max nor min of two intervals need be equal to either of them suggests a different way of thinking about max and min. Thus, instead of defining the max and min of two real numbers by cases as one or the other of them, they are defined, in effect, as the sup and inf of the set consisting of both of them.

The final section is about completeness and limits. Having defined the real number system and established many of its basic properties in §3, we can now repeat for real intervals, more or less *verbatim,* almost everything we did for rational ones. Thus, the counterpart to a real number is a set of real intervals, every two of which intersect and among which are ones of arbitrarily small length. Completeness says that, for each such 'number-like' set of real intervals, $\mathbf{\Lambda}$, there is a unique real number, $\mathbf{lim}\ \mathbf{\Lambda}$, that belongs to every one of them. The definition of $\mathbf{lim}\ \mathbf{\Lambda}$ is elegant: we simply replace each interval in $\mathbf{\Lambda}$ by all rational intervals that contain it and note that consistency and fineness follow directly from the consistency and fineness of $\mathbf{\Lambda}$. The **lim** operator preserves arithmetic and order. (E.g., the limit of a product is the product of the limit.) The chapter concludes with an indication that the rationals are incomplete and a promise to make good on this in the next chapter.

A note on terminology.

The real number system will be defined in the third section of this chapter. Until then, by "a number," we mean a rational number, m/n, and by "an interval," we mean a closed interval, $[r, s]$, with rational endpoints. However, we do not always use these abbreviations.

1. Interval Order and Consistency

Let $I \equiv [r, s]$ and $J \equiv [u, v]$.

1.1. Interval order. $I < J : s < u$.

1.2. Interval consistency. $I \sim J : I$ and J intersect.

Interval trichotomy. Intervals satisfy a trichotomy law that generalizes the one that says that, for all pairs of numbers, x and y, one and only one of the following three relations holds: $x < y$ or $x = y$ or $y < x$.

To formulate the generalization, note first that the intersection of I and J is $[max(r, u),\ min(s, v)]$. Hence, I and J intersect if and only if $max(r, u) \leq min(s, v)$. If we now apply this observation with 1.1 and 1.2 as our interval generalizations of $x < y$ and $x = y$, the following trichotomy law for intervals is an immediate consequence.

1.3. Trichotomy law. Exactly one of the following three relations holds. $I < J$ or $I \sim J$ or $J < I$.

1.4. Weak order. $I \lesssim J :\ I < J$ or $I \sim J$.

Remark. $[r, s] \lesssim [u, v] \iff r \lesssim v \iff p \lesssim q$, for some $p \in [r, s]$ and $q \in [u, v]$.

1.5. $I \lesssim J \iff I \not> J$.

1.6. $I \backsim J \iff I \lesssim J \lesssim I$.

Numbers as intervals of length zero. The definitions above are not just analogous to those for numerical order and equality but actually generalize them in the sense that, for all numbers, x and y, $[x, x] < [y, y] \iff x < y$ and $[x,\ x] \sim [y,\ y] \iff x = y$.

1.7. Transitivity of order. If $I < J < K$, then $I < K$.

Comment. Although 1.7 follows directly from and generalizes the numerical law, $x < y < z \implies x < z$, it is trivial to produce two intervals that intersect a third one but do not intersect each other. Hence, the interval generalization of $x = y = z \implies x = z$ does not hold. Similarly, it is easy to produce intervals for which $I < J \backsim K$

but $I \not< K$. So, the interval generalization of $x < y = z \implies x < z$ also does not hold. Nevertheless, the real number generalizations of these laws do hold and, to show this, it will suffice to prove the following two propositions about intervals.

1.8. If $I < J$, then for all sufficiently short K with $J \lesssim K$, $I < K$.

1.9. If arbitrarily short intervals intersect both I and J, then $I \sim J$.

We also have the following, initially less obvious, relationship, which will be used, later on, to prove transitivity of order for reals.

1.10. A tie that binds. If $I < J \lesssim K < L$, then $I < L$.

Proof. It suffices to consider the case $I < J \sim K < L$. It follows from the definition of the order relation that, for any interval, H, if $I < J \supset H$, then $I < H$ and, if $H \subset K < L$, then $H < L$. But, because $J \sim K$, $J \cap K$ is a subinterval of both J and K. Therefore, $I < J \cap K < L$. Hence, by transitivity, $I < L$.

Note. Although the interval generalization of $x < y = z \implies x < z$ does not hold, we do have the weaker relation, $I < J \backsim K \implies I \lesssim K$.

Max and min. For numbers, $max(x, y)$ can be defined by cases as: if $y \leq x$,then $max(x, y) \equiv x$, else y. But for intervals, I and J, we want to have $max(x, y) \in max(I,\ J)$ for all $x \in I,\ y \in J$ and it is easy to construct examples for which this condition fails for the definition by cases: if $J \lesssim I$,then $max(I, J) \equiv I$, else J.

1.11. Definition. For $I \equiv [r, s],\ J \equiv [u, v]$ and $m \equiv max$ or min, $m(I, J) \equiv [m(r, u), m(s, v)]$.

1.12. $m(I, J)$ contains all $m(x, y)$, for $x \in I$ and $y \in J$.

Proof. Because $m(x, y)$, is a weakly increasing function of each variable, if x is between r and s and y is between u and v, $m(x, y)$ is between $m(r, u)$ and $m(s, v)$.

A least interval property. Because the endpoints of $m(I, J)$ are of the form, $m(x, y)$, for $x \in I$ and $y \in J$, any interval that contains all numbers of this form contains these endpoints and hence the entire interval, $m(I, J)$. In this sense, $m(I, J)$ is the least interval containing all such numbers.

Here are a few more properties of the m operators.

1.13. $I \cap J = max(I, J) \cap min(I, J)$.

1.14. $max(I, J) \backsim J \iff min(I, J) \backsim I \equiv I \lesssim J$.

1.15. $max(I, J) \subset I$ or $max(I, J) \subset J$.

1.16. $max(I, J) < K \iff I < K$ and $J < K$.

1.17. $K < max(I, J) \iff K < I$ or $K < J$.

1.18. We leave it to the reader to formulate and prove analogous necessary and sufficient conditions for $min(I, J) < K$ and $K < min(I, J)$.

Finally, we introduce an interval generalization of the numerical relation, $\mathbf{z}$ is between $\mathbf{x}$ and $\mathbf{y}$.

1.19. *K is **between** I and J* : $min(I, J) \lesssim K \lesssim max(I, J)$

1.20. K is between I and $J \iff I \lesssim K \lesssim J$ or $J \lesssim K \lesssim I$.

1.21. If K is between I and J, and L is between J and K,then L is between I and J.

1.22. If, for each n, $I(n + 2)$ is between $I(n)$ and $I(n + 1)$, then, for all $m > n$, $I(m)$ is between $I(n)$ and $I(n + 1)$.

2. Interval Arithmetic

We now generalize the arithmetic operations on numbers (as well as *magnitude*, which, like *max* and *min*, is not considered an arithmetic operation) to ones on intervals.

2.1. Sum. $I + J \equiv [r + u, s + v]$.

2.2. Difference. $I - J \equiv [r - v, s - u]$. $-J \equiv [-v, -u]$.

2.3. Product. $IJ \equiv [min(ru, rv, su, sv), max(ru, rv, su, sv)]$.

2.4. Magnitude. $|I| \equiv max(I, -I, 0)$. (So, $|-I| = |I|$.)

2.5. Quotient. *For* $|J| > 0$, $1/J \equiv [1/v, 1/u]$ and $I/J \equiv I(1/J)$.

2.6. For all J, $- - J = J$ and if $|J| > 0$, then also $1/1/J = J$.

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2.7. The sum is associative, $(I + J) + K = I + (J + K)$, and commutative, $I + J = J + I$, with respect to the relation, $I = J$, and, hence, also with respect to the weaker one, $I \backsim J$. Likewise, for the product, we have.$I(JK) = (IJ)K$ and $IJ = JI$.

2.8. When $I > 0$, the definition of IJ simplifies. If also $J > 0$, then $IJ = [ru, sv]$. If, instead, $J \backsim 0$, then $IJ = [su, sv]$. Finally, if $J < 0$, then $IJ = [su, rv]$.

2.9. Theorem. If $x \epsilon I$ and $y \epsilon J$, then $x + y \epsilon I + J$, $xy \epsilon IJ$, $x - y \epsilon I - J$, $|x| \in |I|$ and, if $|J| > 0$, $x/y \epsilon I/J$.

A least interval property. Each of the intervals, $I + J$, IJ, $I - J$, $|I|$ and, if $|J| > 0, I/J$, is contained in every interval that contains, respectively, all $x + y$, xy, $x - y$, $|x|$ and x/y for $x \in I$ and $y \in J$.

To prove this, it suffices to check that the endpoints of each of these intervals are of the appropriate kind; e.g., that the endpoints of IJ are of the form, xy, for some $x \in I$ and $y \in J$, etc..

Proof of theorem 2.9. We consider only product and magnitude. The others cases are similar and even simpler.

Product: If $r \leq x \leq s$, then xy is between ry and sy. Likewise, if y is between u and v, then ry is between ru and rv, and sy is between su and sv. Therefore, if $x \epsilon I$ and $y \epsilon J$, then xy lies between the min and max of the four products of an endpoint of I with an endpoint of J, which means that it belongs to the interval, IJ.

Magnitude: If $0 \notin I$, the result is obvious. If $0 \in I$ and $r \leq x \leq s$, then $|x| \in [0, max(-r, s)] = |I|$.

2.10. Corollary. If $I \sim K$ and $J \sim L$, then the value of each operation at I or I and J is consistent with its value at K or K and L. E.g., $I + J \sim K + L$, $IJ \sim KL$, $|I| \backsim |K|$. etc..

This statement follows directly from theorem 2.9, except for case of the reciprocal, for which we require the additional fact that if $J \sim L$ and $|JL| > 0$. then $JL > 0$.

Some laws relating arithmetic, consistency and order. Although the set-theoretic relations of inclusion, $I \subset J$, and equality,

$I = J$, are of interest in their own right, for the role that intervals play in defining and determining properties of real numbers (3.12), their significance is mainly that they imply the much weaker but, for these purposes, more useful relation of intersection, $I \backsim J$.

2.11. If $I \backsim \mathbf{I}$, $J \backsim \mathbf{J}$ and $K \backsim \mathbf{K} \backsim \mathbf{L}$, then $(I + J)K \sim \mathbf{IK} + \mathbf{JL}$. (distributive law)

Proof. If $x \in I \cap \mathbf{I}$, $y \in J \cap \mathbf{J}$ and $z \in K \cap \mathbf{K} \cap \mathbf{L}$, then $(x + y)z = xz + yz$ is in both $(I + J)K$ and $\mathbf{IK} + \mathbf{JL}$..

2.12. $I \backsim J \iff I - J \backsim 0$

2.13. If $|J| > 0$ then $I \sim J \iff I/J \sim 1$.

2.14. Corollary. If $|J| > 0$, then $IJ \backsim 1 \iff I \backsim (1/J)$.

2.15. $I > 0$ and $J > 0 \implies I + J > 0$ and $IJ > 0$.

2.16. $I + J > 0 \implies I > 0$ or $J > 0$.

2.17. $IJ > 0 \iff (I > 0$ and $J > 0)$ or $(I < 0$ and $J < 0)$..

2.18. $IJ \sim 0 \iff I \sim 0$ or $J \sim 0$.

2.19. If $I > J$, then for all sufficiently small $\varepsilon > 0$, $I > J + \varepsilon$.

2.20. If, for all $\varepsilon > 0$, $I \lesssim J + \varepsilon$, then $I \lesssim J$.

In terms of endpoints, 2.20 says that if $r \leq v + \varepsilon$ for all $\varepsilon > 0$,then $r \leq v$, which is precisely the theorem about numbers of which this theorem about intervals is its generalization.

2.21. Definition. The **length** of $I \equiv [r, s]$, is $\mathcal{L}(I) \equiv s - r$.

Remark. For numbers, if $x > 0$, then, for all y, $x + y > y$. By contrast, if $I > 0$, then, for all sufficiently long J, $I + J \backsim J$. However, we do have the following weaker interval version of this numerical law of which the corresponding law for reals is a corollary.

2.22. If $I > J$, then for all sufficiently short K, $I + K > J + K$.

Proof. If $I \equiv [r, s]$ and $J \equiv [u, v]$, then $I > J \iff r > v$. If also $K \equiv [w, z]$, then $I + K > J + K \iff r + w > v + z \iff \mathcal{L}(K) < r - v$.

Thus, for addition, sufficiently short means of length less than $I - J$. Multiplication is more complicated. Not only do we require $K \equiv$

$[w, z] > 0$.but, for any given $I > J$,.the closer w is to 0, the shorter K has to be. We therefore impose a positive lower bound on K.

2.23. If $I > J$ and $c > 0$ then, for all sufficiently short $K > c$, $IK > JK$.

Proof. We reason by cases.

If $I > J \gtrsim 0$, then $IK > JK \iff rw > vz \iff r > vz/w$, .which holds for all $w > c$ and sufficiently small $z - w > 0$.

If $I > 0 > J$, then $IK > 0 > JK$ without any restriction on $K > 0$.

Finally, if $0 \gtrsim I > J$, we can reason as above, replacing I by $-J$ and J by $-I$.

This covers all cases.

2.24. Cancellation laws. Equally important are the cancellation laws, $I + K > J + K \implies I > J$ and $IK > JK \implies I > J$, the latter for $K > 0$. They follow trivially from the definitions with no other restrictions on K. However, this lack of restriction is somewhat illusory because, in the first case, the hypothesis is satisfied only if $\mathcal{L}(K) < I - J$ and, in the second, only if (v/w)(length of K) $< I - J$. Such laws can be used to prove the corresponding cancellation laws for real numbers (3.35, 3.36). However, for second case, the distributive law for intervals provides a simpler way of doing this.

Finally, here are a few useful observations about magnitude.

2.25. $|I| \subset max(I, -I)$.

2.26. $I > 0 \implies |I| = I$. $I < 0 \implies |I| = -I$.

2.27. $I \sim 0 \iff |I| = [0, max(s, -r)]$

Proof. $I \backsim 0$ if and only if $|I| = [max(r, -s, 0), max(-r, s, 0)] = [0, max(-r, s)]$. (But note that, whereas $max(r, -s, 0) = 0$ is equivalent to $r \le 0 \le s$, the relation, $max(-r, s, 0) = max(-r, s)$, holds for all $r \le s$.)

Length Estimates. The simple estimates that we present here, or ones very like them, for the arithmetic operations, max and magnitude are among the pillars on which analysis rests.

2.28. Definitions. Recall that the length of $I \equiv [r, s]$, is defined to be $\mathcal{L}(I) \equiv s - r$. Let $I \equiv [r, s]$ and $J \equiv [u, v]$.

2.29. Sum. $\mathcal{L}(I + J) = \mathcal{L}(I) + \mathcal{L}(J)$.

In terms of endpoints, $s + v - (r + u) = (s - r) + (v - u)$.

2.30. Difference. $\mathcal{L}(I - J) = \mathcal{L}(I) + \mathcal{L}(J)$.

2.31. Product. If $I \subset [-A, A]$ and $J \subset [-B, B]$, then $\mathcal{L}(IJ) \leq B\mathcal{L}(I) + A\mathcal{L}(J)$.

Proof. $IJ = [ab, cd]$ where a and c are the endpoints of I, and b and d are the endpoints of J. Therefore,

$$\mathcal{L}(IJ) = cd - ab = d(c - a) + a(d - b) \leq B\mathcal{L}(I) + A\mathcal{L}(J).$$

2.32. We can also express $\mathcal{L}(IJ)$ symmetrically as the sum of $\mathcal{L}(I)$ times the midpoint of J and $\mathcal{L}(J)$ times the midpoint of I.

2.33. Reciprocal. If $0 < c \leq u$, then $\mathcal{L}(1/J) \leq \mathcal{L}(J)/c^2$.

Proof. $1/u - 1/v = (v - u)/uv$.

2.34. Max. $\mathcal{L}(max(I, J)) \leq max(\mathcal{L}(I), \mathcal{L}(J))$.

2.35. Magnitude. $\mathcal{L}(|I|) \leq \mathcal{L}(I)$.

Proof. $|\, I \,| \subset I$ or $|I| \subset -I$.

3. Real Numbers

Sets of intervals. Let $\mathbf{p}$ and $\mathbf{q}$ be sets of intervals. (Until the next section, by an interval we continue to mean a closed rational interval; i.e., one of the form, $[r, s]$, with $r \leq s$ rational.)

Consistency and order.

3.1. $\mathbf{p} \sim \mathbf{q}$: $I \sim J$, for all $I \in \mathbf{p}$ and $J \in \mathbf{q}$.

3.2. $\mathbf{p} < \mathbf{q}$: $I < J$, for some $I \in \mathbf{p}$ and $J \in \mathbf{q}$.

3.3. $\mathbf{p} \leq \mathbf{q}$: $I \lesssim J$, for all $I \in \mathbf{p}$ and $J \in \mathbf{q}$.

Arithmetic operations.

3.4. Sum. $\mathbf{p} + \mathbf{q} \equiv \{I + J : I \in \mathbf{p},\ J \in \mathbf{q}\}$.

3.5. Difference. $\mathbf{p} - \mathbf{q} \equiv \{I - J : I \in \mathbf{p},\ J \in \mathbf{q}\}$.

3.6. Product. $\mathbf{pq} \equiv \{IJ : I \in \mathbf{p},\ J \in \mathbf{q}\}$.

3.7. Magnitude. $|\mathbf{p}| \equiv \{|I| : I \in \mathbf{p}\}$.

3.8. Reciprocal. If $0 < \mathbf{p}$, then $1/\mathbf{p} \equiv \{1/I : 0 < I \in \mathbf{p}\}$.

3.9. Max. $max(\mathbf{p}, \mathbf{q}) \equiv \{max(I, J) : I \in \mathbf{p},\ J \in \mathbf{q}\}$

The definitions of the binary operations, sum, product and max, extend by induction to finitely many variables.. Thus, $max(\mathbf{p}, \mathbf{q}, \mathbf{w}) \equiv max(max(\mathbf{p}, \mathbf{q}), \mathbf{w})$, etc..

Some special kinds of sets of intervals.

3.10. $\mathbf{p}$ is **consistent**: $\mathbf{p} \backsim \mathbf{p}$.

3.11. $\mathbf{p}$ is **fine**: $\mathbf{p}$ contains intervals of arbitrarily short length.

3.12. $\mathbf{p}$ is a **real number** : $\mathbf{p}$ is consistent and fine: $\mathbf{p} \in \mathbf{R}$.

3.13. $\mathbf{p} = \mathbf{q}$ in $\mathbf{R}$: $\mathbf{p}$ and $\mathbf{q}$ are in $\mathbf{R}$ and $\mathbf{p} \backsim \mathbf{q}$..

Rationals as reals: notation, Even when we treat a whole number, like 2, as a special kind of rational, we usually write '2,' not '2/1'.Similarly, when we treat a rational number, like 1/2, as a special kind of real number, we usually write '1/2,' not anything like $\{[1/2, 1/2]\}$. More generally, any sign we use to denote a rational number, may also be used to denote the associated real.

3.14. A union of equals. The union of all real numbers equal to a given one, $\mathbf{p}$, is itself a real number that is equal to $\mathbf{p}$. It also can be described as the set of all rational intervals that, when considered as real ones, contain $\mathbf{p}$.

The point of this exercise. Although our ultimate object of study is the real number system, the definitions and assertions in this section are formulated either for arbitrary sets of intervals or under the assumption that certain ones are consistent and others are fine. However, the point of considering these less restricted sets of intervals is not to generalize the real number system but, rather, to provide insight into its workings by taking it apart at the "molecular" level.

Some laws of order and consistency.

3.15. Evidently, for all $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{p} \sim \mathbf{q} \iff \mathbf{q} \sim \mathbf{p}$. But not every $\mathbf{p}$ satisfies $\mathbf{p} \backsim \mathbf{p}$. Moreover, although for intervals, $I < J$ obviously implies $I \lesssim J$, it is easy to define sets of intervals, $\mathbf{p}$ and $\mathbf{q}$, for which we have $\mathbf{p} < \mathbf{q}$ but not $\mathbf{p} \leq \mathbf{q}$. Nevertheless, the numerical law, $\mathbf{x} < \mathbf{y} \implies \mathbf{x} \leq \mathbf{y}$, generalizes not only to intervals but to consistent sets of them.

3.16. If $\mathbf{p}$ and $\mathbf{q}$ are consistent, then $\mathbf{p} < \mathbf{q} \implies \mathbf{p} \leq \mathbf{q}$.

3.17. $\mathbf{p} \leq \mathbf{q} \iff r \leq v$ for all $[r, s] \in \mathbf{p}$ and $[u, v] \in \mathbf{q}$.

3.18. $\mathbf{p} \leq \mathbf{q} \iff \mathbf{p} \ngtr \mathbf{q}$.

3.19. $\mathbf{p} \leq \mathbf{q} \leq \mathbf{p} \implies \mathbf{p} \backsim \mathbf{q}$

3.20. If $\mathbf{p} < \mathbf{w} < \mathbf{q}$ and $\mathbf{w}$ is consistent, then $\mathbf{p} < \mathbf{q}$.

3.21. If $\mathbf{p} < \mathbf{w} \leq \mathbf{q}$ and $\mathbf{q}$ is fine, then $\mathbf{p} < \mathbf{q}$.

3.22. Corollary. If $\mathbf{p} \leq \mathbf{w} \leq \mathbf{q}$ and $\mathbf{w}$ is fine, then $\mathbf{p} \leq \mathbf{q}$.

Proof. If $\mathbf{p} > \mathbf{q}$, then by 3.21, $\mathbf{p} > \mathbf{w}$. But $\mathbf{p} \ngtr \mathbf{w}$.

3.23. Corollary of 3.22. If $\mathbf{p} \backsim \mathbf{w} \backsim \mathbf{q}$ and $\mathbf{w}$ is fine, then $\mathbf{p} \backsim \mathbf{q}$.

3.24. If $\mathbf{p} > \mathbf{q}$, then for $\varepsilon > 0$ sufficiently small, $\mathbf{p} > \mathbf{q} + \boldsymbol{\varepsilon}$.

This follows directly from the analogous result for intervals.

3.25. Corollary. If, for all $\varepsilon > 0$, $\mathbf{p} \leq \mathbf{q}+ \varepsilon$, then $\mathbf{p} \leq \mathbf{q}$.

3.26. If $\mathbf{p}$ and $\mathbf{q}$ are fine and there are arbitrarily short $I \ni \mathbf{p}$ and $J \ni \mathbf{q}$.with $I \lesssim J$, then $\mathbf{p} \leq \mathbf{q}$.

3.27. ε-Trichotomy. If $\mathbf{p}$ and $\mathbf{q}$ are fine, then for all $\varepsilon > 0$, $\mathbf{p} < \mathbf{q}$, $\mathbf{q} < \mathbf{p}$ or $|\mathbf{p} - \mathbf{q}| < \varepsilon$.

Proof. By interval trichotomy, given any $I \in \mathbf{p}$, $J \in \mathbf{q}$, either (1) $I < J$, in which case $\mathbf{p} < \mathbf{q}$, or (2) $J < I$,in which case $\mathbf{q} < \mathbf{p}$, or

(3) $I \sim J$, in which case $|\mathbf{p} - \mathbf{q}| \leq \mathcal{L}(I) + \mathcal{L}(J)$. And because $\mathbf{p}$ and $\mathbf{q}$ are fine, for each $\varepsilon > 0$, there are intervals, $I \in \mathbf{p}$, $J \in \mathbf{q}$, with $\mathcal{L}(I) + \mathcal{L}(J) < \varepsilon$.

Some laws of arithmetic.

3.28. It follows directly from 2.7 that the operations of sum and product for sets of intervals are associative and commutative with respect to the relation, $\mathbf{p} \backsim \mathbf{q}$. Similarly, the distributive law for sets of intervals follows directly from the one for intervals.

3.29. The distributive law. $\mathbf{w}(\mathbf{p} + \mathbf{q}) = \mathbf{wp} + \mathbf{wq}$.

3.30. Theorem. If $\mathbf{p}$ and $\mathbf{q}$ are real numbers, so are $\mathbf{p} + \mathbf{q}$, $\mathbf{p} - \mathbf{q}$, $\mathbf{pq}$, $max(\mathbf{p}, \mathbf{q})$, $|\mathbf{p}|$ and, if $|\mathbf{p}| > \mathbf{0}$, $\mathbf{1}/\mathbf{p}$.

Proof. It suffices to show that these operations preserve both consistency and fineness.

Lemma 1. If $\mathbf{p}$ and $\mathbf{q}$ are consistent, so are $\mathbf{p} + \mathbf{q}$, $\mathbf{p} - \mathbf{q}$, $\mathbf{pq}$, $max(\mathbf{p}, \mathbf{q})$, $|\mathbf{p}|$ and, if $|\mathbf{p}| > \mathbf{0}$, $\mathbf{1}/\mathbf{p}$.

Lemma 2. If $\mathbf{p}$ and $\mathbf{q}$ are fine, so are $\mathbf{p} + \mathbf{q}$, $\mathbf{p} - \mathbf{q}$, $\mathbf{pq}$, $max(\mathbf{p}, \mathbf{q})$, $|\mathbf{p}|$ and, if $|\mathbf{p}| > \mathbf{0}$, $\mathbf{1}/\mathbf{p}$.

Proof of lemmas. The first is an immediate consequence of our consistency results for interval arithmetic, magnitude and max. As for the second, here we will consider only the product and the reciprocal. The other cases are simpler because, for them, the length estimate we seek depends only on the lengths of the intervals to which the operation being considered is applied, not on the size of the numbers they contain.

Reciprocals. Given $\mathbf{p} > 0$, there is a $c \in \mathbf{Q}$ such that $\mathbf{p} \geq c > 0$. Hence, for all $I \in \mathbf{p}$ of sufficiently small length, we have $I > c$ and $\mathcal{L}(1/I) = (s - r)/rs \leq \mathcal{L}(I)/c^2$. Therefore, to have $\mathcal{L}(1/I) \leq \varepsilon$, it suffices to take an I so small that $\mathcal{L}(I) \leq \varepsilon c^2$.

Products. If $|\mathbf{p}| < A$ and $|\mathbf{q}| < B$, then all small enough intervals, $I \in \mathbf{p}$ and $J \in \mathbf{q}$, are contained respectively in $[-A, A]$ and $[-B, B]$. Then, for all such I and J, $\mathcal{L}(IJ) \leq B\mathcal{L}(I) + A\mathcal{L}(J)$. Hence, for all $\varepsilon > 0$, $\mathcal{L}(IJ) \leq \varepsilon$ provided that both $\mathcal{L}(I)$ and $\mathcal{L}(J)$ are $\leq \varepsilon/(A + B)$.

3.31. Equals to equals. By 2.10, each of these operations preserves the relation, $I \backsim J$, for intervals and, hence, also, for arbitrary sets of them. I.e., if $\mathbf{p} \backsim \mathbf{w}$ and $\mathbf{q} \backsim \mathbf{z}$, then the value of each of these operations at $\mathbf{p}$ and $\mathbf{q}$ is consistent with its value at $\mathbf{w}$ and $\mathbf{z}$. Hence, because, by 3.28, each of these operations sends $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{R}$ to a value that is also in $\mathbf{R}$,it follows that they send equals in $\mathbf{R}$ to equals in $\mathbf{R}$.

Some laws relating order and arithmetic.

3.32. If $\mathbf{p} > \mathbf{0}$ and $\mathbf{q} > \mathbf{0}$, then $\mathbf{p} + \mathbf{q} > \mathbf{0}$ and $\mathbf{pq} > \mathbf{0}$.

3.33. If $\mathbf{p} + \mathbf{q} > \mathbf{0}$, then $\mathbf{p} > \mathbf{0}$ or $\mathbf{q} > \mathbf{0}$.

3.34. If $\mathbf{pq} > \mathbf{0}$, then $(\mathbf{p} > \mathbf{0}$ and $\mathbf{q} > \mathbf{0})$ or $(\mathbf{p} < \mathbf{0}$ and $\mathbf{q} < \mathbf{0})$..

3.35. If $\mathbf{p} > \mathbf{q}$ and $\mathbf{w}$ is fine, then $\mathbf{p} + \mathbf{w} > \mathbf{q} + \mathbf{w}$.

3.36. Corollary. If $\mathbf{p} + \mathbf{w} \lesssim \mathbf{q} + \mathbf{w}$ and $\mathbf{w}$ is fine, then $\mathbf{p} \lesssim \mathbf{q}$.

3.37. If $\mathbf{p} + \mathbf{w} > \mathbf{q} + \mathbf{w}$, then $\mathbf{p} > \mathbf{q}$

3.38. If $\mathbf{p} \backsim \mathbf{q}$, then $\mathbf{p} + \mathbf{w} \backsim \mathbf{q} + \mathbf{w}$.

Note. 3.38 follows trivially from 3.37 but even more trivially from its interval counterpart, which itself follows trivially from the definitions.

3.39. If $\mathbf{p} > \mathbf{q}$ and $\mathbf{w} > \mathbf{0}$ is fine, then $\mathbf{pw} > \mathbf{qw}$.

3.40. Corollary. If $\mathbf{pw} \backsim \mathbf{qw}$ and $\mathbf{w} > \mathbf{0}$ is fine, then $\mathbf{p} \backsim \mathbf{q}$

3.41. If $\mathbf{pw} > \mathbf{qw}$ and $\mathbf{w} > \mathbf{0}$, then $\mathbf{p} > \mathbf{q}$

.

3.42. If $\mathbf{p} \backsim \mathbf{q}$ and $\mathbf{w} > \mathbf{0}$, then $\mathbf{pw} \backsim \mathbf{qw}$.

3.43. If $\mathbf{q} > \mathbf{0}$ then $\mathbf{p} < \mathbf{q} \iff \mathbf{p}/\mathbf{q} < \mathbf{1}$.

3.44. If $\mathbf{q} > \mathbf{0}$, then $\mathbf{p} \backsim \mathbf{q} \iff \mathbf{p}/\mathbf{q} \backsim \mathbf{1}$..

As with 3.38 and 3.37, 3.44 follows trivially from 3.43 but even more trivially from its interval counterpart.

3.45. $max(\mathbf{p}, \mathbf{q}) > \mathbf{w} \iff \mathbf{p} > \mathbf{w}$ or $\mathbf{q} > \mathbf{w}$.

3.46. Corollary. If $max(\mathbf{p}, \mathbf{q}) \backsim \mathbf{q}$, then $\mathbf{p} \leq \mathbf{q}$. (Set $\mathbf{w} \equiv \mathbf{q}$ in 3.45.)

3.47. $max(\mathbf{p}, \mathbf{q}) < \mathbf{w} \iff \mathbf{p} < \mathbf{w}$ and $\mathbf{q} < \mathbf{w}$.

3.48. Definition. For $S \subset \mathbf{R}$ and $\mathbf{p} \in \mathbf{R}$, $\mathbf{p} = sup\ S : \mathbf{p} \geq \mathbf{s}$ for all $\mathbf{s} \in S$ and, for all $\mathbf{q} < \mathbf{p}$ in $\mathbf{R}$, there exists an $\mathbf{s} \in S$ with $\mathbf{s} > \mathbf{q}$.

3.49. Uniqueness. If $\mathbf{p} = sup\ S = \mathbf{q}$, then $\mathbf{p} = \mathbf{q}$.

Proof. If $\mathbf{q} < \mathbf{p} = sup\ S$, there exists $\mathbf{s} \in S$ with $\mathbf{q} < \mathbf{s}$. But if $\mathbf{q} = sup\ S$, then $\mathbf{s} \leq \mathbf{q}$. Hence, if $\mathbf{p} = sup\ S = \mathbf{q}$,then $\mathbf{p} \not> \mathbf{q}$ and, for the same reason, $\mathbf{q} \not> \mathbf{p}$. Hence, $\mathbf{p} = \mathbf{q}$.

3.50. An abuse of notation. Notice that we have not defined a real number named '$sup\ S$' to which every $\mathbf{p}$ satisfying definition 3.43 is equal. Thus, it would be more accurate to let this expression denote the set of all such $\mathbf{p}$, so that, instead of writing $\mathbf{p} = sup\ S$, we would write $\mathbf{p} \in sup\ S$. However, because all such $\mathbf{p}$ are equal in $\mathbf{R}$, this abuse of notation seems harmless.

3.51. For $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{R}$, $max(\mathbf{p}, \mathbf{q}) = sup\{\mathbf{p}, \mathbf{q}\}$.

Proof. We have to show that, for each $\varepsilon > 0$, $max(\mathbf{p}, \mathbf{q}) - \mathbf{w} < \varepsilon$ for either $\mathbf{w} \equiv \mathbf{p}$ or $\mathbf{w} \equiv \mathbf{q}$. To this end, take intervals, $I \in \mathbf{p}$ and $J \in \mathbf{q}$, of length $< \varepsilon$ and recall that $max(I, J)$ is contained in either I or J.

4. Completeness and Limits

The definitions of 'fine' and 'consistent' extend verbatim to any set of closed real intervals, $\mathbf{\Lambda}$.

4.1. Definition. $\mathbf{\Lambda}$ is number-like : $\mathbf{\Lambda}$ is fine and consistent.

4.2. Definition. $\mathbf{Lim}\ \mathbf{\Lambda} \equiv \{[r, s] : r, s$ rational and some $[p, q] \in \Lambda$ is contained in $[r, s]\}$.

4.3. Completeness theorem. If $\mathbf{\Lambda}$ is number-like, then $\mathbf{Lim}\ \mathbf{\Lambda}$ is a real number that belongs to every member of $\mathbf{\Lambda}$. Moreover, every real number that belongs to every member of $\mathbf{\Lambda}$ is equal to $\mathbf{Lim}\ \mathbf{\Lambda}$.

Proof. Because each pair of intervals in $\mathbf{\Lambda}$ intersect, so does every pair of rational intervals each of which contains a member of $\mathbf{\Lambda}$. Hence, $\mathbf{Lim}\ \mathbf{\Lambda}$ is consistent. The proof that $\mathbf{Lim}\ \mathbf{\Lambda}$ is fine is by 'rational approximation.' The idea is that so long as we can get arbitrarily good real approximations to something, we can get arbitrarily good rational ones by first approximating by a real to a better accuracy than we seek and then approximating that real by a rational to within the difference.

Here is a formal rendering of this. Given $\in > 0$, choose positive rationals, α, β and γ such that $\alpha + \beta + \gamma \leq \in$. Because $\mathbf{\Lambda}$ is fine, there is an interval, $I \equiv [p, q] \in \mathbf{\Lambda}$ with $\mathcal{L}(I) \leq \alpha$. Because p and q are fine, there exist $J \equiv [u, v] \in p$ and $K \equiv [w, z] \in q$ such that $0 \leq p - u \leq \mathcal{L}(J) \leq \beta$ and $0 \leq z - q \leq \mathcal{L}(K) \leq \gamma$. Then $[u, z]$ is a rational interval of length at most ε that contains $[p, q]$.

To see that $I \in \mathbf{\Lambda} \implies \mathbf{Lim}\ \mathbf{\Lambda} \in I$, note that every $J \in \mathbf{Lim}\ \mathbf{\Lambda}$ intersects every $I \in \mathbf{\Lambda}$. Finally, because there are arbitrarily small intervals in $\mathbf{\Lambda}$, if both $\mathbf{Lim}\ \mathbf{\Lambda}$ and $\mathbf{p}$ are in every one of them, then $\mathbf{p} = \mathbf{Lim}\ \mathbf{\Lambda}$. (It suffices to recall that two real numbers that differ by less than every $\varepsilon > 0$ are equal.)

Arithmetic and limits. The section on the order and arithmetic of consistent, fine sets of rational intervals (a.k.a. ‘real numbers’) can be repeated nearly verbatim for consistent, fine (a.k.a. ‘number-like’) sets of real ones. We first define the order and the arithmetic operations (also $max(I, J)$ and $|I|$) for real intervals much as we did for rational ones. Then, in essentially the same way, we extend them to arbitrary number-like families.

4.4. Theorem. If Λ and Γ are number-like, so are $\Lambda + \Gamma$, $\Lambda - \Gamma$, $\Lambda\Gamma$, if $|\Gamma| > 0$, Λ/Γ, $max(\Lambda, \Gamma)$ and $|\Lambda|$.

The basic truth of the arithmetic of limits for number-like sets is that each of these operations "commutes" with the **lim** operator in the following sense.

4.5. Theorem. If Λ and Γ are number-like, then

$$\mathbf{lim}(\Lambda + \Gamma) = \mathbf{lim}\ \Lambda + \mathbf{lim}\ \Gamma,$$

$$\mathbf{lim}(\Lambda - \Gamma) = \mathbf{lim}\ \Lambda - \mathbf{lim}\ \Gamma,$$

$$\mathbf{lim}\ (\Lambda\Gamma) = \mathbf{lim}\ \Lambda\ \mathbf{lim}\ \Gamma,$$

$$\mathbf{lim}\ (\Lambda/\Gamma) = \mathbf{lim}\ \Lambda/\mathbf{lim}\ \Gamma \text{ if } |\Gamma| > 0,$$

$$\mathbf{lim}\ max(\mathbf{\Lambda}, \mathbf{\Gamma}) = max(\mathbf{lim}\ \mathbf{\Lambda}, \mathbf{lim}\ \Gamma),$$

$$\mathbf{lim}\ |I| = |\mathbf{lim}\ I|.$$

Order and limits. Adding and removing **lim** are order-preserving operations.

4.6. Theorem. $\mathbf{\Lambda} > \mathbf{\Gamma} \iff \mathbf{lim}(\mathbf{\Lambda}) > \mathbf{lim}(\mathbf{\Gamma})$.

4.7. Corollary. $\mathbf{\Lambda} \leq \mathbf{\Gamma} \iff \mathbf{lim}(\mathbf{\Lambda}) \leq \mathbf{lim}(\mathbf{\Gamma})$.

How the reals complete the rationals. Many problems that are formulated in terms of integers do not admit an integer solution but do have a rational one. E.g., $2x = 1$. So, we can and do use rational numbers to find out things about integers. Likewise, many problems formulated in terms of rationals do not have a rational solution but do have a real one—which, being a real number and, hence, fine, can be approximated by rationals to arbitrary accuracy. Here is a simple example.

4.8. Definition. $a^{m/n} = b : a^m = b^n$.

4.9. Theorem. There is no solution to the equation $2^{m/n} = 3$, i.e., to $2^m = 3^n$.

Proof. For positive integers, m and n, 2^m is always even and 3^n is always odd.

But, in the next chapter, we define the exponential function, a^x,for all $a \geq 1$ and show that, for $b > 0$, the equation $a^x = b$ has a real number solution, x, that satisfies $a^m = b^n$ if x is equal to a rational number, m/n. It follows from the theorem above that, for $2^x = 3$, this solution, which is denoted $log_2 3$, is not rational. (Nevertheless, there is a simple procedure, essentially trial and error, for obtaining arbitrarily small rational intervals containing it. For example, $2^1 < 3^1 < 2^2 \implies log_2 3 \in [1, 2]$. So we try $3/2 :\ 2^3 > 3^2 \implies log_2 3 \in [3/2, 2]$. Etc..)

CHAPTER 2

An Inverse Function Theorem and Some Applications

(This is chapter 2 of *A New Course of Analysis: The Main Ideas*, by Gabriel Stolzenberg. Updated January, 2006.)

In chapter 1, we were concerned with real numbers in general but none in particular. In this one, we first apply some of the general truths that we have learned about the real number system—above all, about the **lim** operator—to obtain a simple but powerful inverse function theorem. By succesive applications of it, we derive substantial results about exponentials and logarithms. Here too, our arguments are not traditional ones. The latter, which require integration, differentiation and power series, will be presented in later chapters.

More precisely, in §2, bounds for $|y^n - x^n|/|y - x|$ away from 0 and ∞ are fed into the inverse function theorem, yielding $x^{1/n}$ and, more generally, x^r for $r \in \mathbf{Q}$. This is straightfoward but what happens next is less so.

In §3, we use these bounds again—but in a different way—to get ones for $(A^s - A^r)/(s - r)$ for $A > 1$ and r, $s \in \mathbf{Q}$. These, in turn, enable us to use the **lim** operator to define A^x for $A > 1$, $x \in \mathbf{R}$ and the arithmetic of limits to extend the bounds to real exponents. Another application of the inverse function theorem yields $log_A x$ for $A > 1$.

Finally, in §4, we construct a function, $L(a)$, that is the derivative of a^x at $x = 0$. However, at this stage, it does not help to know what a derivative is. We demonstrate that $L(a^\lambda) = \lambda L(a)$ for all $\lambda > 0$, $a^{1/L(a)} = b^{1/L(b)}$ for all a, $b > 1$, and define $e \equiv 2^{1/L(2)}$. It follows that $L(a) = log_e(a)$.

1. An Inverse Function Theorem

1.1. Theorem. Suppose $f : [a,b] \longrightarrow \mathbf{R}$. If, for some constants, $0 < L \leq K$ and all $x \leq y$ in $[a, b]$, $L\,(y- x) \leq f(y) - f(x) \leq K\,(y - x)$, then there exists $g : [f(a),\ f(b)] \longrightarrow [a,b]$ such that, for all $w \geq\ z$ in $[f(a), f(b)]$, $f(g(z)) = z$ and $(w - z)/K \leq g(w) - g(z) \leq (w - z)/L$.

The plan. In applications, we often know that $a < b$. However, the construction of g does not require knowing either this or that $a = b$. For each $z \in [f(a),\ f(b)]$, we define a sequence of intervals, $I_n(z) \equiv [a_n, b_n] \subset I_0(z) \equiv [a,b]$, in such a way that $z \epsilon [f(a_n),\ f(b_n)]$, $I_{n+1}(z) \subset I_n(z)$ and, for some C independent of n, $b_n - a_n \leq C/2^n$. The second and third conditions imply that $\{I_n(z) : n \in N\}$ is number-like. We define $g(z)$ to be its limit and check that it has the required properties.

Proof. Set $I_0(z) \equiv [a,b]$. The proof of ε-trichotomy (3.27) gives a procedure that, for each k, yields either $a < b$ or $|b - a| \leq 1/2^k$. For each n, if, for all $k \leq n$, it yields $|b-a| \leq 1/2^k$, we define $I_n(z) \equiv [a,b]$ and apply the procedure for $n+1$. If this yields $|b - a| \leq 1/2^{n+1}$, we continue with $n+1$ in place of n.

But if, instead, it yields $a < b$, then, for all $m \geq n$, $I_{m+1}(z)$ will be defined to be either the left, right or middle half of $I_m(z)$. I.e., if $I_m(z) \equiv [A,B]$ and M, P and Q denote the midpoints of $[A,B]$, $[A,M]$ and $[M,B]$, then $I_{m+1}(z)$ will be either $[A,M]$, $[M,B]$ or $[P,Q]$.

To decide which of these three intervals to choose, note first that, because $B - A > 0$, so is $m \equiv min(f(Q) - f(M),\ f(M) - f(P)) \geq L(B - A)/4$.. Hence, there are rational intervals, I and J, for z and $f(M)$, respectively, such that $\boldsymbol{\mathcal{L}}(I) + \boldsymbol{\mathcal{L}}(J) \leq m \equiv min(f(Q) - f(M), f(M) - f(P))$. We now use I and J to decide which of the three intervals to choose for $I_{m+1}(z)$.

If $I < J$, z is between $f(A)$ and $f(M)$ and we set $I_{m+1}(z) \equiv [A,M]$. Similarly, if $I > J$, z is between $f(M)$ and $f(B)$ and we set $I_{m+1}(z) \equiv [M,B]$. Finally, if $I \sim J$, z is between $f(M) - (\boldsymbol{\mathcal{L}}(I) + \boldsymbol{\mathcal{L}}(J))$ and $f(M) + (\boldsymbol{\mathcal{L}}(I) + \boldsymbol{\mathcal{L}}(J))$ and hence also between $f(P)$ and $f(Q)$. In this case, we set $I_{m+1}(z) \equiv [P,Q]$.

Then $\{I_n(z)\}$ is a number-like set of intervals and $g(z) \equiv \mathbf{Lim}\{I_n(z)\}$ is a real number that belongs to every $I_n(z) \equiv [a_n, b_n]$. Hence, $f(g(z))$ is in each $[f(a_n), f(b_n)]$. As is z. But $f(b_n) - f(a_n) \leq K(b_n - a_n) \leq C/2^n$

for $C \equiv Kmax((b-a),1)$. So, for all $\varepsilon > 0$, $|f(g(z)) - z| \leq \varepsilon$. Hence, $f(g(z)) = z$.

Moreover, for all $z \leq w$ in $[f(a),\ f(b)]$, $g(z) \leq g(w)$ in $[a,b]$. So, it follows from the bounds for $f(y) - f(x)$ for $x \leq y$ in $[a,b]$ that $w - z$ is between $L\,(g(w) - g(z))$ and $K\,(g(w) - g(z))$. Hence, $g(w) - g(z)$ is between $(w-z)/K$ and $(w-z)/L$.

1.2. Corollary. For all $x \in [a,b]$, $g(f(x)) = x$.

Proof. Note that if $|x - g(f(x))| > 0$, then $|f(x) - f(x)| > 0$.

2. Powers and Roots

In this section, we derive inequalities for $y^n - x^n$ that enable us to apply the inverse function theorem to define the nth root, $x^{1/n}$. In the next section, we use these inequalities to get ones for $A^s - A^r$, for $A \geq 1$ and rational exponents, r and s. These, in turn, enable to use completeness to define A^x, for $A \geq 1$ and x real. We then use the arithmetic of limits to extend these inequalities from rational exponents to real ones. This enables us to apply the inverse function theorem to define, for $A > 1, log_A x$, the inverse of A^x. In the final section, we use a comparison of $A^x - 1$ with x to construct the real number, e, and derive some of its more important properties.

2.1. Proposition. For $x,\ y \in \mathbf{R}$ and $n \in \mathbf{N}$,

$$y^n - x^n = \left(\sum_{i=0}^{n-1} y^i x^{n-1-i}\right)(y-x).$$

Proof. By induction, starting with
$y^{n+1} - x^{n+1} = y^{n+1} - xy^n + xy^n - x^{n+1} = y^n(y-x) + x(y^n - x^n)$.

2.2. Corollary. If $x,\ y > 0$, then $x^n < y^n \iff x < y$ and $x^n = y^n \iff x = y$.

2.3. Corollary. If $a \geq 0$,then, for all $x \leq y$ in $[a,b]$,

$$na^{n-1}(y-x) \leq y^n - x^n \leq nb^{n-1}(y-x).$$

2.4. Corollary. There is a function, $z^{1/n}$, on $[a^n, b^n]$ such that, for all z and w in $[a^n, b^n]$,

$$(z^{1/n})^n = z \text{ and } (w-z)/nb^{n-1} \leq w^{1/n} - z^{1/n} \leq (w-z)/na^{n-1}.$$

Proof. To apply the Inverse Function Theorem, we need to know that each z is contained in some $[a^n, b^n]$ with $a > 0$. So we need to know that, for all positive integers, n, the function, $f(x) \equiv x^n$ assumes both arbitrarily small and arbitrarily large values on $(0, \infty)$. Which is obvious because, for all $x > 1$ and $n \geq 1$, $(1/x)^n \leq 1/x \leq x \leq x^n$.

2.5. Corollary. If $x, y > 0$, then $x^{1/n} < y^{1/n} \iff x < y$.

2.6. Corollary of the corollary. For $x, y > 0$, $x^{1/n} = y^{1/n} \iff x = y$.

2.7. Definition. For $r \equiv m/n$, $x^{-r} \equiv 1/x^r$.

3. Exponentials and Logarithms

3.1. Definition. $A^{m/n} \equiv (A^m)^{1/n}$.

3.2. Laws of exponents. $A^{m/n} = (A^{1/n})^m$. $(AB)^{m/n} = A^{m/n}B^{m/n}$. $A^{(p+q)/n} = A^{p/n}A^{q/n}$.

Proof. For the first law, check that the nth powers of $(A^{1/n})^m$ and $(A^{m)1/n}$ are equal. For the second, use $(xy)^n = x^n y^n$. For the third law, use the first to get

$$A^{(p+q)/n} = (A^{1/n})^{p+q} = (A^{1/n})^p(A^{1/n})^q = A^{p/n}A^{q/n}.$$

3.3. Lemma. If $A \geq 1$ and $r \leq s$ in $\mathbf{Q}$, then $A^r \leq A^s$.

Proof. Write $A^s - A^r = A^r(A^{m/n} - 1)$ for $m/n \equiv s - r \geq 0$.

Then $A^r \leq A^s \iff 1 \leq A^{m/n}$. But $A \geq 1 \implies A^m \geq 1 \implies A^{m/n} \geq 1$.

3.4. Proposition. For $A \geq 1$ and $r \leq s$ in $\mathbf{Q}$,

$$A^s - A^r \leq A^s(A - 1)(s - r).$$

Proof. Using the notation of the proof of 3.3, by 2.3 for $y^m - x^m$ with $x \equiv 1$ and $y \equiv A^{1/n}$, we have $m(A^{1/n} - 1) \leq A^{m/n} - 1$ and

$$A^{m/n} - 1 \leq m(A^{1/n})^{m-1}(A^{1/n} - 1) \leq m(A^{1/n})^m(A^{1/n} - 1).$$

Hence, $m(A^{1/n} - 1) \leq A^{m/n} - 1 \leq m(A^{1/n})^m(A^{1/n} - 1)$.

For $m = n$, the lower inequality yields $A^{1/n} - 1 \leq (A - 1)/n$, and this, together with the upper one, yields $A^{m/n} - 1 \leq (m/n)A^{m/n}(A - 1)$.

If we now replace m/n by $s-r$ and multiply through by A^r, we get the inequality asserted in the proposition..

Note. Similar reasoning yields $A^{r-1}(A-1)(s-r) \leq A^s - A^r$.

3.5. Definition. For $A \geq 1$ and $x \in \mathbf{R}$,

$$\sigma(A, x) \equiv \{[A^r, A^s] : [r, s] \in x\}.$$

3.6. Theorem. $\sigma(A, x)$ is number-like.

Proof. Because x is consistent, if $[r, s]$ and $[u, v]$ are both x-intervals, then $r \leq v$ and $u \leq s$. Hence, by the preceding lemma, $A^r \leq A^v$ and $A^u \leq A^s$. Thus, $\sigma(A, x)$ is consistent. To see that it also is fine, note first that, by the lemmas above, $0 \leq A^s - A^r \leq A^s(A-1)(s-r)$. It follows from this that, for any $\delta > 0$ and all $[r, s] \in x$ with $s - r < \delta$, we have $s < x + \delta$ and $|A^s - A^r| \leq A^{x+\delta}(A-1)(s-r)$. Therefore, $\sigma(A, x)$ is fine.

3.7. Definition. For $A \geq 1$ and $x \in \mathbf{R}$, $A^x \equiv \mathbf{Lim}\ \sigma(A, x)$.

3.8. Laws of real exponents.

1. If $A \geq 1$, then, for all x and y, $A^{x+y} = A^x A^y$.
2. If also $B \geq 1$, then, for all x, $(AB)^x = A^x B^x$.
3. Finally, if also $x \geq 0$, then, for all y, $(A^x)^y = A^{xy}$.

3.9. The restriction to $x \geq 0$ is needed to insure that $B^x \geq 1$, so that we can form $(B^x)^y$. In the next section, we extend the definition of A^x, together with the laws of exponents, to all $A > 0$ and all $x \in \mathbf{R}$.

3.10. To verify these laws, we can follow the method of proof of the next theorem. E.g., for the first law, we express A^{x+y} and $A^x A^y$ as limits of families of intervals, $\sigma(A, x+y)$ and $\sigma(A, x)\sigma(A, y)$, and show that, for all $I \in \sigma(A, x+y)$ and $J \in \sigma(A, x)\sigma(A, y)$, $I \sim J$.

An Inverse for $\mathbf{A}^x$. The following theorem provides upper and lower bounds for $A^y - A^x$ that enable us to apply the inverse function theorem (1.1) to construct an inverse for A^x for $A > 1$. As for its proof, we verified the upper inequality for rational exponents in the course of showing that $\sigma(A, x)$ is fine. The lower inequality can be demonstrated much the same way and, as we show below, we can then use rational approximation to show that, in the limit, the inequalities hold for real exponents too.

3.11. Theorem. For $A \geq 1$ and $x \leq y$ in $\mathbf{R}$,

$$A^{x-1}(A-1)(y-x) \leq A^y - A^x \leq A^y(A-1)(y-x).$$

Proof. By the arithmetic of limits, each of the terms in the chain of inequalities above is a limit of one the following three sets of intervals (the first of the first, the second of the second and the third of the third).

$$\{[A^{r-1}, A^{s-1}](A-1\ )[u-s, v-r] : [r,s] \in x, [u,v] \in y\},$$

$$\{[A^u - A^s, A^v - A^r] : [r,s] \in x, [u,v] \in y\} \text{ and}$$

$$\{[A^u,\ A^v](A-1\ )[u-s, v-r] : [r,s] \in x, [u,v] \in y\}.$$

By the proof of 3.6, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $[r,s] \in x$ and $[u,v] \in y$ are both of length $< \delta$, then the lengths of the associated intervals, one from each of these three sets, are all $< \varepsilon$.

Hence, because weak inequalities are preserved in the limit (4.7), it suffices to show that, for each $[r,s] \in x$ and $[u,v] \in y$, the three asociated intervals satisfy the relevant chain of inequalities.

For this, by 1.6, it suffices to show that there are real numbers, $\mathbf{p}$ in the first interval, $\mathbf{q}$ in the second and $\mathbf{w}$ in the third, with $\mathbf{p} \leq \mathbf{q} \leq \mathbf{w}$. Which there are, because, by proposition 3.4, $A^{r-1}(A-1)(v-r) \leq A^v - A^r \leq A^v(A-1)(v-r)$. So we are done.

3.12. Remark. Given these inequalities for $A > 1$,we can now apply the inverse function theorem to get an inverse for A^x on any interval of the form, $[A^c,\ A^d]$. It remains to show that this inverse is defined on all of $(0, \infty)$.

3.13. Lemma. If $a \equiv A - 1 > 0$, then, for all positive integers, n, $1 + na \leq A^n$.

3.14. Corollary. If $A > 1$,then every $\mathbf{p} > \mathbf{0}$ belongs to an interval of the form, $[A^c, A^d]$.

Proof. Because $a \equiv A-1 > 0$, we can take the d in the statement to be any integer n so large that $p \leq 1 + na$. For c, we turn the inequality upside down, getting $A^{-n} \leq 1/(1+na)$, which means that we can take c to be $-n$ for any n so large that $1/(1+na) \leq p$.

3.15. Definition. For $A \in (1, \infty)$, $log_A(x) \equiv$ the inverse of A^x on $(0, \infty)$.

4. The Natural Logarithm and the Euler Number, e

By theorem 3.11, for $a \geq 1$ and $0 < |x| < 1$, $(a^x - 1)/x$ is between $(a-1)/a$ and $a(a-1)$. Here we will show that there is a real number, $L(a)$,to which $(a^x - 1)/x$ becomes arbitrarily close for x sufficiently small.

4.1. Definition. For $a \geq 1$,

$\mathbf{\Lambda}(a) \equiv \{[(a^y - 1)/y, (a^x - 1)/x] : y < 0 < x\}$.

4.2. Theorem. $\mathbf{\Lambda}(a)$ is number-like.

Proof. To demonstrate that $\mathbf{\Lambda}(a)$ is consistent, we note first that $A^z - A^w \leq A^z(A-1)(z-w)$ for all $0 \leq w < z$.

Substituting $w = 0$ gives $A^z - 1 \leq A^z(A-1)z$. Setting $A \equiv a^x \geq 1$ for $x > 0$,we get $(a^x)^z = a^{xz}$ and hence $(a^{xz} - 1)/z \leq a^{xz}(a^x - 1)$.

Finally, after substituting y for $-xz < 0$ with $x > 0$ and dividing by a^y, we get that, for all $y < 0 < x$, $(a^{-y} - 1)/(-y/x) \ \leq a^{-y}(a^x - 1)$ and hence $(a^y - 1)/y \leq (a^x - 1)/x$.

To demonsrate that $\mathbf{\Lambda}(a)$ is fine, we estimate of the length of those $\mathbf{\Lambda}(a)$-intervals for which $y = -x$.

$$\frac{a^x - 1}{x} - \frac{a^x - 1}{xa^x} = \frac{(a^x - 1)^2}{xa^x} \leq \frac{x^2 a^{2x}(a-1)^2}{xa^x} = xa^x(a-1)^2.$$

Thus, for any $d > 0$, $C \equiv a^d(a-1)^2$.and $0 < x \leq d$,the length of the $\Lambda(a)$-interval with endpoints, $y = -x$ and x is $\leq \ Cx$.

4.3. Definition. For $a \geq 1$, $L(a) \equiv \mathbf{lim}\ \Lambda(a)$.

4.4. Theorem. For all $\lambda > 0$, $L(a^\lambda) = \lambda L(a)$.

Proof. Because $\lambda > 0$, $a^\lambda \geq 1$. Hence, $(a^\lambda)^x$ is defined. For $X = \lambda x$, $((a^\lambda)^x - 1)/x = \lambda(a^{\lambda x} - 1)/\lambda x = (a^X - 1)/X$,which implies that an interval J is in $\Lambda(a^\lambda)$ if and only if $J = \lambda I$ for some $I \in \Lambda(a)$. Therefore, passing to the limit, we have

$L(a^\lambda) \equiv \mathbf{lim}\Lambda(a^\lambda) = \mathbf{lim}\lambda\Lambda(a) = \lambda\mathbf{lim}\ \Lambda(a) = \lambda L(a).$

4.5. Proposition. $L(1) = 0.$ and, for $a > 1$, $L(a) > 0$.

Proof. By the lower bound given by 3.11 for $a^y - a^x$ with $x = 0$, $L(a) \geq (a^y - 1)/y \geq (a-1)/a > 0$.

The next proposition uses the relationship $b = a^\lambda$ for $a, b > 1$ and $\lambda \equiv log_a b$.

4.6. Proposition. For $a, b > 1$, $L(ab) = L(a) + L(b)$ and $a^{L(b)} = b^{L(a)}$.

Proof. First, $L(ab) = L(a^{1+\lambda}) = L(a) + \lambda L(a) = L(a) + L(b)$. Next, $a^\lambda > 1$ because $a > 1$ and $\lambda > 0$. Hence, $(a^\lambda)^{L(a)}$ is defined and we can write

$$a^{L(b)} = a^{L(a^\lambda)} = a^{\lambda L(a)} = (a^\lambda)^{L(a)} = b^{L(a)}.$$

4.7. Corollary. For all $a,\ b > 1$, $a^{1/L(a)} = b^{1/L(b)}$.

4.8. Definition. The Euler number, $e \equiv 2^{1/L(2)}$. So, $L(e) = 1$.

4.9. Corollary. For all $a \geq 1$, $e^{L(a)} = a$. I.e., $L(a) = log_e a$.

4.8. Proposition. For all $a \geq 1$, $a^x = 2^{x log_2(a)}$.

4.10. Remark. Recall that $log_2 a$ is defined for all $a > 0$. Using this and the preceding observation, we can extend definition of the exponential function from $a \geq 1$ to $a > 0$.

4.11. Definition. For all $a > 0$ and $x \in \mathbf{R}$, $a^x \equiv 2^{x log_2(a)}$.

4.12. Laws of exponents. For all $a, b > 0$, $(a^x)^y = a^{xy}$, $a^{x+y} = a^x a^y$ and $(ab)^x = a^x b^x$.

Proofs. Exercise.

4.13. Proposition. For all $x > 0$, $e^x \geq 1 + x$.

Proof. $1 = L(e) \leq (e^x - 1)/x$.

Comparing e with 3: For $a > 1$, each $\Lambda(a)$-interval gives upper and lower bounds for $L(a)$ and, hence, also for $e = a^{1/L(a)}$. E.g., $a \equiv 2$ and $x \equiv 1$ gives $1/2 \leq L(2) \leq 1$. Therefore, $e = 2^{1/L(2)}$ is between 2 and 4. Let's try next to compare it with 3.

Note first that $e < 3 \iff 3^{1/L(3)} < 3 \iff 1 < L(3)$. So, recalling the definition of $L(3)$, it suffices to find an $x > 0$ for which $3^x < (3^x - 1)/x$. To keep the computations simple, let us look first for an x of the form, $1/n$. In that case, we seek an integer, $n > 0$ for which $n < 3^{1/n}(n-1)$,which is equivalent to having $n^n < 3(n-1)^n$.

So, for example, $n \equiv 5$ fails because $5^5 = 3125 > 3 \times 4^5 = 3 \times 2^{10} = 3072$. But $n \equiv 6$ works: $(6/5)^6 = (216/125)^2 = (1.728)^2 < (1.73)^2 = 2.9929 < 3$. So $2 < e < 3$.