

# Rigorous justification of Taylor dispersion via center manifolds and hypocoercivity

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## Abstract

Taylor diffusion (or dispersion) refers to a phenomenon discovered experimentally by Taylor in the 1950s where a solute dropped into a pipe with a background shear flow experiences diffusion at a rate proportional to  $1/\nu$ , which is much faster than what would be produced by the static fluid if its viscosity is  $0 < \nu \ll 1$ . This phenomenon is analyzed rigorously using the linear PDE governing the evolution of the solute. It is shown that the solution can be split into two pieces, an approximate solution and a remainder term. The approximate solution is governed by an infinite-dimensional system of ODEs that possesses a finite-dimensional center manifold, on which the dynamics correspond to diffusion at a rate proportional to  $1/\nu$ . The remainder term is shown to decay at a rate that is much faster than the leading order behavior of the approximate solution. This is proven using a spectral decomposition in Fourier space and a hypocoercive estimate to control the intermediate Fourier modes.

## 1 Introduction

Taylor dispersion is a phenomenon in fluid dynamics that was discovered in the 1950's by Geoffrey Taylor [Tay53, Tay54]. The setting is a three dimensional pipe in which there is a background shear flow advecting the fluid down the length of the pipe, but where the rate of advection can vary as a function of the cross-sectional variables. It was observed by Taylor that, if a localized drop of dye was put into the pipe, then as expected it would be carried down the pipe by the shear flow and also diffuse due to the non-zero fluid viscosity. However, what was not expected was that the rate of diffusion experienced by the dye was not that of the fluid, say  $\nu$ , but instead a rate proportional to  $1/\nu$ , which is much larger if  $0 < \nu \ll 1$ . This phenomenon has been subsequently analyzed by many people, for example [Ari56, CA85, MR90], but most of the work has been formal, based on asymptotic calculations. Our goal in this work is to rigorously analyze Taylor dispersion and provide a mathematical mechanism for its occurrence using center manifolds and Villani's theory of hypocoercivity [Vil09]. We note there is another rigorous analysis of Taylor dispersion, [BCZ17], that also uses hypocoercivity in the proof. We will comment on the relationship between that and the present work at the end of this section.

The PDE model of fluid flow in a pipe with a background shear flow is given by

$$u_t = \nu \Delta u - V(y, z)u_x, \quad x \in \mathbb{R}, \quad (y, z) \in \Omega \subset \mathbb{R}^2.$$

The function  $u : \mathbb{R} \times \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$  represents the concentration of the solute, or dye, and the function  $V : \Omega \rightarrow \mathbb{R}$  is a smooth background shear flow, which depends only on the cross-sectional variables  $(y, z) \in \Omega$ , where  $\Omega$  is compact with smooth boundary. We assume Neumann boundary conditions,

$$\frac{\partial u}{\partial n} \Big|_{\partial \Omega} = 0.$$

For simplicity we assume the viscosity is a small, positive constant,  $0 < \nu \ll 1$ . To remove any effects of constant background advection caused by  $V$ , we define  $\chi$  via

$$V(y, z) = A(1 + \chi(y, z)), \quad A = \frac{1}{\text{vol}(\Omega)} \int_{\Omega} V(y, z) dy dz,$$

and require that  $\chi \in H^2(\Omega)$ . Thus,  $A$  is the average rate of advection in a cross section, and  $\chi$  therefore has zero average advection in a cross section. We can then change variables using  $x \rightarrow x + At$  to obtain

$$u_t = \nu \Delta u - A\chi(y, z)u_x. \tag{1.1}$$

It will be convenient to separate the effects of the cross-stream and longitudinal pipe variables. To that end, we will expand both  $u$  and  $\chi$  in terms of the eigenfunctions of the Laplacian  $\partial_y^2 + \partial_z^2$  acting on the compact domain  $\Omega$ . These eigenfunctions, which we denote by  $\{\psi_n\}_{n=0}^\infty$ , form an orthonormal basis for  $L^2(\Omega)$  with  $\psi_0 \equiv 1$ , and we denote their corresponding eigenvalues by  $\{-\mu_n\}_{n=0}^\infty$ , which satisfy  $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$  [Str08, §11.3]. It will also be helpful to scale the longitudinal space variable  $x$  and the time variable  $t$  by  $\nu$  via

$$X = \nu x, \quad T = \nu t. \quad (1.2)$$

This transforms (1.1) into

$$u_T = \nu^2 u_{XX} + \Delta_{y,z} u - A\chi(y, z)u_X. \quad (1.3)$$

The main advantage of this change is that it helps us determine the dependence of the solutions on the viscosity parameter  $\nu \ll 1$ . This advantage will be made clear in Remarks 3.3 and 4.9. Inserting the expansions

$$u(X, y, z, T) = \sum_{n=0}^{\infty} u_n(X, T)\psi_n(y, z), \quad \chi(y, z) = \sum_{n=0}^{\infty} \chi_n \psi_n(y, z), \quad (1.4)$$

where

$$u_n(X, T) = \int_{\Omega} u(X, y, z, T)\psi_n(y, z)dydz, \quad \chi_n = \int_{\Omega} \chi(y, z)\psi_n(y, z)dydz,$$

into equation (1.3) and noting that  $\chi_0 = 0$  since it has zero average in  $\Omega$ , we obtain

$$\partial_T u_0 = \nu^2 \partial_X^2 u_0 - A \sum_{m=1}^{\infty} \chi_m \partial_X u_m \quad (1.5)$$

$$\partial_T u_n = \nu^2 \partial_X^2 u_n - \mu_n u_n - A\chi_n \partial_X u_0 - A \sum_{m=1}^{\infty} \chi_{n,m} \partial_X u_m, \quad n = 1, 2, \dots, \quad (1.6)$$

where

$$\chi_{n,m} = \langle \psi_n, \chi \psi_m \rangle_{L^2(\Omega)}.$$

In order to use invariant manifolds to study Taylor dispersion, we must deal with the fact that the Laplacian,  $\partial_X^2$ , on  $\mathbb{R}$  has continuous spectrum consisting of  $(-\infty, 0]$ ; in other words, there is no spectral gap. One way to overcome this is to use similarity variables,

$$\xi = \frac{X}{\sqrt{T+1}}, \quad \tau = \log(T+1),$$

which exploit the space/time scaling inherent to the operator [Way97]. (The use of  $T+1$ , rather than  $T$ , in the above definition is just for convenience, so that the change of variables is well-defined at  $T=0$ .)

We therefore further define new dependent variables  $\{w_n\}_{n=0}^\infty$  via

$$u_0(X, T) = \frac{1}{\sqrt{T+1}} w_0 \left( \frac{X}{\sqrt{T+1}}, \log(T+1) \right) \quad (1.7)$$

$$u_n(X, T) = \frac{1}{T+1} w_n \left( \frac{X}{\sqrt{T+1}}, \log(T+1) \right), \quad n = 1, 2, \dots \quad (1.8)$$

Plugging this definition into (1.5)-(1.6), we obtain

$$\partial_\tau w_0 = \mathcal{L}w_0 - A \sum_{m=1}^{\infty} \chi_m \partial_\xi w_m \quad (1.9)$$

$$\partial_\tau w_n = \left( \mathcal{L} + \frac{1}{2} \right) w_n - e^{\tau/2} A \sum_{m=1}^{\infty} \chi_{n,m} \partial_\xi w_m - e^\tau (\mu_n w_n + A \chi_n \partial_\xi w_0), \quad (1.10)$$

where

$$\mathcal{L} = \nu^2 \partial_\xi^2 + \frac{1}{2} \partial_\xi (\xi \cdot) = \nu^2 \partial_\xi^2 + \frac{1}{2} \xi \partial_\xi + \frac{1}{2} \quad (1.11)$$

is the Laplacian  $\nu^2 \partial_x^2$  written in terms of the similarity variables. Note that the reason for the different powers of  $(T+1)$  in front of  $w_0$  and  $w_n$  for  $n \geq 1$  in (1.7) is that equation (1.9) above becomes  $\tau$ -independent. Continuing, we remark that the operator  $\mathcal{L}$  was analyzed in detail in [GW02]. Its properties are given in §2 below, but for the moment we just note that, on the space

$$L^2(m) = \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + \xi^2)^m |w(\xi)|^2 d\xi < \infty \right\}, \quad (1.12)$$

the spectrum of  $\mathcal{L}$  is composed of essential and discrete spectrum:

$$\sigma(\mathcal{L}) = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq -(2m-1)/4 \} \cup \{ \lambda = -k/2 : k = 0, 1, 2, \dots \}.$$

Thus, as the algebraic weight  $m$  in the definition of the function space  $L^2(m)$  increases, the essential spectrum is pushed further into the left half-plane, revealing more and more isolated eigenvalues at negative multiples of  $1/2$ . This suggests that we can construct a center-stable manifold (which we often refer to as a center manifold, for short) corresponding to those isolated eigenvalues, where the dimension of this manifold can be large if  $m$  is sufficiently large.

The utility of such a center manifold can be seen by considering the term  $-e^\tau (\mu_n w_n + A \chi_n \partial_\xi w_0)$  in (1.10). As  $\tau$  increases this term becomes large, which suggests that  $w_n$  should evolve so that ultimately  $\mu_n w_n + A \chi_n \partial_\xi w_0 = 0$ . Hence, we expect that, for large times,

$$w_n \approx -\frac{A \chi_n}{\mu_n} \partial_\xi w_0 \quad \Rightarrow \quad \partial_\tau w_0 \approx \mathcal{L}_{td} w_0,$$

where

$$\mathcal{L}_{td} := (\nu^2 + A^2 \|\chi\|_\mu^2) \partial_\xi^2 + \frac{1}{2} \partial_\xi (\xi \cdot), \quad \|\chi\|_\mu^2 = \sum_m \frac{1}{\mu_m} \chi_m^2 \quad (1.13)$$

is again the Laplacian in similarity variables but now with Taylor diffusion coefficient

$$\nu_{td} := (\nu^2 + A^2 \|\chi\|_\mu^2). \quad (1.14)$$

Note that the spectrum of the operator does not depend on the viscosity, so  $\sigma(\mathcal{L}) = \sigma(\mathcal{L}_{td})$ . Thus, we expect that  $\{w_n\}_{n=1}^\infty$  will rapidly converge to a manifold defined by  $w_n = -(A \chi_n \partial_\xi w_0) / (\mu_n)$ , and then for large times the dynamics of  $w_0$  can be described by a center-stable manifold corresponding to the isolated eigenvalues of the operator  $\mathcal{L}_{td}$ . In terms of the original variables, this suggests that  $\{u_n\}_{n=1}^\infty$  should become “slaved” to the low mode  $u_0$  exponentially fast, while the low mode  $u_0$  should decay diffusively, but as if its diffusion coefficient is  $\nu_{td} = \mathcal{O}(1)$  (instead of  $\nu^2$ ), which, if we change back to the original  $(x, t)$  variables, matches the experimental observations of Taylor and the formal calculations in [CA85].

There are several technical difficulties that must be overcome in order to make the above argument rigorous. First, in analyzing the dynamics of system (1.9)-(1.10) using the spectral structure of  $\mathcal{L}_{td}$ , it would be natural to expand each  $w_n$ ,  $n = 0, 1, \dots$ , in terms of the eigenfunctions  $\{\varphi_j^{td}(\xi)\}_{j=0}^N$  of  $\mathcal{L}_{td}$ , where  $N = N(m)$  corresponds to the number of isolated eigenvalues, and hence the dimension of the center-stable manifold. In other words, we could write

$$P_N w_n(\xi, \tau) = \sum_{j=0}^N \alpha_{j,n}(\tau) \varphi_j^{td}(\xi), \quad w_n^s = (1 - P_N)w_n$$

for each  $n$ , where  $w_n^s$  is the component of the solution in the strong stable manifold, which we expect to decay rapidly. Although this is essentially what we will do, it turns out that it will be more convenient to prove the rapid decay of  $w_n^s$  in terms of the  $(X, T)$  variables, by using the Fourier transform.

The reason for this is that our center manifold argument will only show that the enhanced diffusion affects the first  $N + 1$  terms in the eigenfunction expansion  $\{w_n\}$ . This is sufficient for the physical realization of the phenomenon because the higher order terms, corresponding to  $w_n^s$ , will be shown to decay like  $T^{-\mathcal{N}(N)}$  where  $\mathcal{N}$  can be made large by choosing  $N$ , and hence also  $m$ , to be large, which is faster than the enhanced algebraic diffusive decay resulting from Taylor diffusion.

To understand what  $P_N w_n$  corresponds to in the physical  $(x, t)$  variables, consider the following calculation. The eigenfunctions of  $\mathcal{L}_{td}$  are given by

$$\varphi_j^{td}(\xi) = \partial_\xi^j \varphi_0^{td}(\xi), \quad \varphi_0^{td}(\xi) = \frac{1}{\sqrt{4\pi\nu_{td}}} e^{-\frac{\xi^2}{4\nu_{td}}}. \quad (1.15)$$

If we assume that

$$u(X, T) = \frac{1}{(1+T)^\gamma} w\left(\frac{X}{\sqrt{T+1}}, \log(T+1)\right), \quad w(\xi, \tau) = \sum_{j=0}^N \alpha_j(\tau) \varphi_j^{td}(\xi),$$

which can represent either  $w_0$  or  $w_n$ ,  $n \geq 1$ , depending on the choice of  $\gamma$ , then

$$\begin{aligned} \hat{u}(\kappa, T) &= \int e^{i\kappa X} u(X, T) dX \\ &= \sum_{j=0}^N \frac{1}{\sqrt{4\pi\nu_{td}}} (1+T)^{j/2-\gamma} (-i\kappa)^j \alpha_j(\log(T+1)) \int e^{i\kappa X} e^{-\frac{X^2}{4\nu_{td}(T+1)}} dX \\ &= \sum_{j=0}^N (1+T)^{j/2+1/2-\gamma} (-i\kappa)^j \alpha_j(\log(T+1)) e^{-\kappa^2 \nu_{td}(T+1)}. \end{aligned}$$

This implies that

$$\hat{u}(0, T) = (1+T)^{1/2-\gamma} \alpha_0(\log(T+1)), \quad \hat{\partial}_\kappa u(0, T) = (-i)(1+T)^{1-\gamma} \alpha_1(\log(T+1)), \quad \dots$$

which, combined with the Taylor expansion

$$\hat{u}(\kappa, T) = \hat{u}(0, T) + \partial_\kappa \hat{u}(0, T) \kappa + \frac{1}{2} \partial_\kappa^2 \hat{u}(0, T) \kappa^2 + \dots$$

means that the behavior of  $P_N w$  tells us about the behavior of  $\hat{u}(\kappa, T)$  for  $\kappa$  near zero. In other words,  $P_N w$  represents both the behavior of the “low modes” of  $w(\xi, \tau)$ , where “low modes” refers to the leading

eigenfunctions of  $\mathcal{L}_{td}$ , and the behavior of the “low modes” of  $\hat{u}(\kappa, T)$ , where now “low modes” refers to values of the Fourier variable  $\kappa$  near zero. This relationship between Taylor dispersion, the behavior of the Fourier transform of the solution at small wave numbers, and the center-manifold theorem was also discussed by Mercer and Roberts in [MR90].

We will refer to  $w^s = (1 - P_N)w$  as the remainder, or error, term. In terms of  $\mathcal{L}_{td}$  it corresponds to the behavior due to the essential spectrum and the discrete spectrum that is sufficiently far from the imaginary axis. To prove that the remainder term decays rapidly, it will be convenient to work in terms of the Fourier variables associated with physical  $(X, T)$  space, rather than system (1.9)-(1.10). This will lead to a linear, nonautonomous equation governing the behavior of the remainder term of the form  $\hat{U}_T = \mathcal{B}(\kappa)\hat{U} + \hat{F}(\kappa, T)$ . We can then consider three regimes: a small wavenumber regime defined by  $|\kappa| \leq \kappa_0$ , an intermediate one defined by  $\kappa_0 \leq |\kappa| \leq \kappa_1/\nu$ , and a large one defined by  $|\kappa| \geq \kappa_1/\nu$ . In the large regime, the solution decays exponentially due to the usual (non-Taylor) diffusive estimate  $e^{\nu^2 \partial_X^2 T} \sim e^{-\nu^2 \kappa^2 T} \leq e^{-\kappa_1^2 T}$ . In the intermediate regime this naive estimate is not quite strong enough, because it only implies  $e^{\nu^2 \partial_X^2 T} \sim e^{-\nu^2 \kappa^2 T} \leq e^{-\nu^2 \kappa_0^2 T}$ , which is quite weak for  $0 < \nu \ll 1$ . To improve it, we will apply a hypocoercivity argument [Vil09] to show that in this region we also have decay like  $e^{-MT}$  for some  $M > 0$ . For the low wavenumbers, we will decompose the remainder term into a piece corresponding to the leading eigenvalue  $\lambda_0(\kappa)$  of  $\mathcal{B}(\kappa)$ , which is parabolic with  $\lambda_0(0) = 0$ , and a piece corresponding to the rest of the spectrum of  $\mathcal{B}(\kappa)$ . The latter will decay exponentially fast because  $\mathcal{B}(k)$  has a spectral gap for each fixed  $k$ . The former will be shown to decay algebraically with the rate  $T^{-\mathcal{N}(N)}$ , because we have already removed the leading order behavior via the term  $P_N w_n$ .

Our analysis will be divided into the following steps. In §2 we will more precisely set-up our problem and carefully state the main results. In §3 we will use the similarity variables and a center-stable manifold to prove that the low modes, corresponding to  $P_N w_n$ , experience enhanced Taylor diffusion. Finally, in §4 we will use a spectral decomposition and hypocoercivity to show that the remainder term decays rapidly, thus allowing for the Taylor diffusion to be physically observable.

Before carrying this out, we comment on other related rigorous work on Taylor diffusion. In [BCW15] we analyzed a model of system (1.5)-(1.6) consisting of only two equations, one corresponding to  $u_0$  and one modeling all of the  $u_n$  for  $n \geq 1$ , and carried out a similar analysis there. This allowed us to focus on the main ideas of the argument: that the Taylor diffusion is really only affecting the low modes, with the remainder term decaying rapidly. However, in that work, because of the simple form of the system, one could see directly that the remainder term decayed rapidly and the hypocoercivity argument we use here in §4 was not necessary. Moreover, the center manifold argument, which was used to justify the enhanced diffusion, was constructed for a finite-dimensional ODE. Here, the center manifold argument in §3 will need to be carried out for an infinite-dimensional ODE.

Also, in [BCZ17] an equation very similar to (1.1) was analyzed, also using hypocoercivity. However, there Villani’s framework was applied directly to the PDE (1.1), whereas our hypocoercivity argument is applied in Fourier space. This allows us to avoid any assumptions on the critical points of the shear flow  $\chi$ , which play an important role in the argument in [BCZ17]. Moreover, since  $X \in \mathbb{R}$ , we need to work in Fourier space with all  $|\kappa| \geq 0$ . The setting in [BCZ17] is for a bounded  $X$  domain, which effectively means  $|\kappa| \geq 1$ . This changes the nature of the resulting decay and the regions in which the enhanced diffusion is obtained.

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## 2 Set-up and statement of main results

The main result that we will prove is the following. Theorem 1(i) will be proven in §3 and Theorem 1(ii) will be proven in §4. In the statement of the Theorem we use the following notation for the space in which the initial data must lie:

$$L^2(N+1) \times L^2(\Omega) = \left\{ u \in L^2(\mathbb{R} \times \Omega) : \int_{\mathbb{R}} \int_{\Omega} (1 + X^2)^{N+1} |u(X, y, z)|^2 dX dy dz =: \|u\|_{L^2(N+1) \times L^2(\Omega)}^2 < \infty \right\}.$$

**Theorem 1.** *Given any  $N > 0$ , if  $u(\cdot, 0) \in L^2(N+1) \times L^2(\Omega)$ , then there exist constants  $C_j = C_j(\|u(\cdot, 0)\|_{L^2(N+1) \times L^2(\Omega)})$ ,  $j = 1, 2$ , that are independent of  $\nu$  and a decomposition of the corresponding solution of (1.3) of the form*

$$u(X, y, z, T) = u_{\text{app}}(X, y, z, T) + u_{\text{rem}}(X, y, z, T),$$

where  $u_{\text{app}}(X, y, z, T)$  and  $u_{\text{rem}}(X, y, z, T)$  are defined in (2.11)-(2.12), that satisfies the following.

(i) *There exists an infinite-dimensional system of ordinary differential equations that govern the behavior of  $u_{\text{app}}$ . Moreover, this system of ODEs possesses a finite dimensional center manifold that is globally attracting at a rate that is exponential in  $T$ ,  $e^{-\eta T}$  for some  $\eta$  independent of  $\nu$ , and on which the dynamics correspond to enhanced diffusion with viscosity  $\nu_{td}$ , defined in (1.14). In other words,*

$$\left\| u_{\text{app}}(X, y, z, T) - \frac{C_1}{\sqrt{4\pi\nu_{td}(T+1)}} e^{-\frac{x^2}{4\nu_{td}(T+1)}} \right\|_{L^2(\mathbb{R} \times \Omega)} \leq \frac{C_2}{(1+T)^{3/4}},$$

where  $C$  is a constant that is independent of  $\nu$ .

(ii) *The remainder term satisfies*

$$\|u_{\text{rem}}(\cdot, T)\|_{L^2(\mathbb{R} \times \Omega)} \leq \frac{C_2}{(1+T)^{\frac{N}{6} + \frac{1}{12}}}.$$

If we translate these results back to our original, unscaled time and space variables and choose  $N \geq 4$ , so that  $\frac{N}{6} + \frac{1}{12} \geq \frac{3}{4}$ , we see that we obtain immediately:

**Corollary 2.1.** *Given any initial condition  $u(\cdot, 0) \in L^2(N+1) \times L^2(\Omega)$ , there exists constants  $C_j = C_j(\|u(\cdot, 0)\|_{L^2(N+1) \times L^2(\Omega)})$ ,  $j = 1, 2$ , such that the solution of (1.1) satisfies*

$$\left\| u(x, y, z, t) - \frac{C_1}{\sqrt{4\pi(\nu + A^2\|\chi\|_{\mu/\nu})(t+1/\nu)}} e^{-\frac{x^2}{4(\nu + A^2\|\chi\|_{\mu/\nu})(t+1/\nu)}} \right\|_{L^2(\mathbb{R} \times \Omega)} \leq \frac{C_2}{(1+\nu t)^{3/4}}.$$

**Remark 2.2.** Note that the leading order term in the asymptotics identified by this Corollary corresponds to a solution of the diffusion equation with diffusion coefficient  $(\nu + A^2\|\chi\|_\mu/\nu)$  which is precisely the asymptotic behavior derived non-rigorously in [Smi87]. (In particular, see (2.17) for the calculation of the shear diffusion coefficient.)

**Remark 2.3.** As we discuss later in Section 3, we actually derive not just the leading order term in the asymptotics but higher terms as well - in principle, terms of arbitrary order, if the initial condition  $u_0$  decays sufficiently rapidly as  $|x| \rightarrow \infty$ . The higher order terms in the asymptotics are expressed in terms of the eigenfunctions of the operator  $\mathcal{L}_{td}$ .

To prove these results, we will use the following facts about the operator  $\mathcal{L}_{td}$ , which is just the Laplacian written in terms of similarity variables. Recall from (1.13) that

$$\mathcal{L}_{td}\varphi = \nu_{td}\partial_\xi^2\varphi + \frac{1}{2}\partial_\xi(\xi\varphi).$$

We state the following results for viscosity  $\nu_{td}$ , but the results are true with  $\nu_{td}$  replaced by any other positive number. This operator has been analyzed in [GW02], and in the weighted Hilbert space  $L^2(m)$  defined in (1.12) one finds

$$\sigma(\mathcal{L}_{td}) = \left\{ \lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq -\frac{(2m-1)}{4} \right\} \cup \left\{ -\frac{k}{2} \mid k \in \mathbb{N} \right\}.$$

Furthermore, the eigenfunctions corresponding to the isolated eigenvalues  $\lambda_k = -k/2$  are given by the Hermite functions

$$\varphi_0^{td}(\xi) = \frac{1}{\sqrt{4\pi\nu_{td}}}e^{-\frac{\xi^2}{4\nu_{td}}}, \quad \varphi_k^{td}(\xi) = \partial_\xi^k \varphi_0^{td}(\xi).$$

The corresponding adjoint eigenfunctions are given by the Hermite polynomials

$$H_k^{td}(\xi) = \frac{2^k \nu_{td}^k}{k!} e^{\frac{\xi^2}{4\nu_{td}}} \partial_\xi^k e^{-\frac{\xi^2}{4\nu_{td}}}. \quad (2.1)$$

Note that we have the orthogonality relationship

$$\langle H_k^{td}, \varphi_j^{td} \rangle_{L^2(\mathbb{R})} = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases}$$

which can be used to define spectral projections.

**Remark 2.4.** The expressions in [GW02] for  $\varphi_k^{td}$  and  $H_k^{td}$  are derived in the case when  $\nu_{td} = 1$ . The expressions given here follow easily by the change of variables  $\xi \rightarrow \xi/\sqrt{\nu_{td}}$ .

## 2.1 Preparation of the equations

To emphasize the expected role of the enhanced diffusion, we rewrite (1.9)-(1.10) as

$$\partial_\tau w_0 = \mathcal{L}_{td}w_0 - D_{td}\partial_\xi^2 w_0 - A \sum_{m=1}^{\infty} \chi_m \partial_\xi w_m \quad (2.2)$$

$$\partial_\tau w_n = \left( \mathcal{L}_{td} + \frac{1}{2} \right) w_n - D_{td}\partial_\xi^2 w_n - e^{\tau/2} A \sum_{m=1}^{\infty} \chi_{n,m} \partial_\xi w_m - e^\tau (\mu_n w_n + A \chi_n \partial_\xi w_0), \quad (2.3)$$



where

$$D_{td} := A^2 \|\chi\|_\mu^2$$

and  $\mathcal{L}_{td}$  is defined in equation (1.13). As described above, asymptotically we expect  $w_n = -(A\chi_n \partial_\xi w_0)/(\mu_n)$ , which is a perfect derivative. To exploit this, we wish to effectively integrate the  $w_n$  equation. Naively, this could be done by defining  $\{V_n\}_{n=1}^\infty$  via  $\partial_\xi V_n = w_n$ . In order to obtain decay of  $V_n$  as  $|\xi| \rightarrow \infty$ , we would then need to assume that  $\int w_n = 0$ . To avoid this additional assumption, we instead define  $\{V_n\}_{n=1}^\infty$  via

$$w_n(\xi, \tau) = \gamma_n(\tau) \varphi_0^{td}(\xi) + V_n(\xi, \tau), \quad \gamma_n(\tau) = \int_{\mathbb{R}} w_n(\xi, \tau) d\xi, \quad (2.4)$$

where  $\varphi_0^{td}$  is the eigenfunction of  $\mathcal{L}_{td}$  defined in (1.15) associated with the zero eigenvalue. Note that this implies

$$\gamma_n(\tau) = \langle w_n(\tau), H_0^{td} \rangle_{L^2} = \int_{\mathbb{R}} w_n(\xi, \tau) d\xi$$

and that  $\gamma_n(\tau)$  is bounded for each  $\tau$  such that  $w_n(\tau) \in L^2(m)$ , with  $m > 1/2$ , because

$$|\gamma_n(\tau)| \leq \int_{\mathbb{R}} \frac{1}{(1 + \xi^2)^{\frac{m}{2}}} (1 + \xi^2)^{\frac{m}{2}} |w(\xi, \tau)| d\xi \leq \left( \int_{\mathbb{R}} \frac{1}{(1 + \xi^2)^m} d\xi \right)^{1/2} \|w_n(\tau)\|_{L^2(m)} \leq C(m) \|w_n(\tau)\|_{L^2(m)}.$$

Since  $\int \varphi_0^{td} = 1$ , we see that  $\int V_n = 0$ . Inserting (2.4) into (2.3), we find

$$\begin{aligned} \dot{\gamma}_n \varphi_0^{td} + \partial_\tau V_n &= \frac{1}{2} \gamma_n \varphi_0^{td} + \left( \mathcal{L}_{td} + \frac{1}{2} \right) V_n - D_{td} (\gamma_n \varphi_2^T + \partial_\xi^2 V_n) \\ &\quad - e^{\tau/2} A \sum_{m=1}^{\infty} \chi_{n,m} \partial_\xi (\gamma_m \varphi_0^{td} + V_m) - e^\tau (\mu_n V_n + A \chi_n \partial_\xi w_0) - e^\tau \mu_n \gamma_n \varphi_0^{td}. \end{aligned} \quad (2.5)$$

Integrating over  $\mathbb{R}$  and using the fact that  $\varphi_k^{td}, w_0 \rightarrow 0$  as  $|\xi| \rightarrow \infty$ , we find

$$\dot{\gamma}_n = \left( \frac{1}{2} - e^\tau \mu_n \right) \gamma_n,$$

which implies that

$$\gamma_n(\tau) = \gamma_n(0) e^{\frac{\tau}{2} - \mu_n (e^\tau - 1)}. \quad (2.6)$$

With this information, in (2.5) we can cancel all the terms involving  $\gamma_n$  alone, use the fact that  $\int V_n = 0$  to define  $v_n$  via  $\partial_\xi v_n = V_n$ , and obtain from (2.2)-(2.3)

$$\partial_\tau w_0 = \mathcal{L}_{td} w_0 - D_{td} \partial_\xi^2 w_0 - A \sum_{m=1}^{\infty} \chi_m \partial_\xi^2 v_m - A e^{\frac{\tau}{2}} \varphi_1^{td} \sum_{m=1}^{\infty} \chi_m \gamma_m(0) e^{-\mu_m (e^\tau - 1)} \quad (2.7)$$

$$\begin{aligned} \partial_\tau v_n &= \mathcal{L}_{td} v_n - D_{td} \partial_\xi^2 v_n - e^{\tau/2} A \sum_{m=1}^{\infty} \chi_{n,m} \partial_\xi v_m - e^\tau (\mu_n v_n + A \chi_n w_0) \\ &\quad - e^\tau A \varphi_0^{td} \sum_{m=1}^{\infty} \chi_{n,m} \gamma_m(0) e^{-\mu_m (e^\tau - 1)} - D_{td} \gamma_n(0) e^{\frac{\tau}{2}} e^{-\mu_n (e^\tau - 1)} \varphi_1^{td}. \end{aligned} \quad (2.8)$$

## 2.2 Separation into low modes and the remainder term

In order to analyze the behavior of solutions to system (2.7)-(2.8), we define

$$\begin{aligned} w_0(\xi, \tau) &= \sum_{k=0}^N \alpha_k(\tau) \varphi_k^{td}(\xi) + w_0^s(\xi, \tau) \\ v_n(\xi, \tau) &= \sum_{k=0}^N \beta_k^n(\tau) \varphi_k^{td}(\xi) + v_n^s(\xi, \tau), \end{aligned} \quad (2.9)$$

where  $\{\varphi_k^{td}\}_{k=0}^N$  are the first  $N+1$  eigenfunctions associated with  $\mathcal{L}_{td}$  and

$$\alpha_k(\tau) = \langle w_0(\xi, \tau), H_k^{td}(\xi) \rangle_{L^2(\mathbb{R})}, \quad \beta_k^n(\tau) = \langle v_n(\xi, \tau), H_k^{td}(\xi) \rangle_{L^2(\mathbb{R})},$$

are the spectral projections onto those eigenmodes defined via the corresponding adjoint eigenfunctions  $H_k^{td}$ . See (2.1). Recalling that  $\partial_\xi \varphi_k^{td} = \varphi_{k+1}^{td}$  and  $\mathcal{L}_{td} \varphi_k^{td} = -(k/2) \varphi_k^{td}$ , inserting the above expressions into (2.7)-(2.8) and taking the inner product of the result with  $H_k^{td}$  gives the following infinite-dimensional system of ODEs for the evolution of  $\{\alpha_k\}_{k=0}^N$  and  $\{\beta_k^n\}_{k=0}^N$ ,  $n \geq 1$ :

$$\begin{aligned} \dot{\alpha}_0 &= 0 \\ \dot{\alpha}_1 &= -\frac{1}{2} \alpha_1 - A e^{\frac{\tau}{2}} \sum_{m=1}^{\infty} \chi_m \gamma_m(0) e^{-\mu_m(e^\tau - 1)} \\ \dot{\alpha}_k &= -\frac{k}{2} \alpha_k - D_{td} \alpha_{k-2} - A \sum_{m=1}^{\infty} \chi_m \beta_{k-2}^m \quad 2 \leq k \leq N \\ \dot{\beta}_0^n &= -e^\tau (\mu_n \beta_0^n + A \chi_n \alpha_0) - e^\tau A \sum_{m=1}^{\infty} \chi_{n,m} \gamma_m(0) e^{-\mu_m(e^\tau - 1)} \\ \dot{\beta}_1^n &= -\frac{1}{2} \beta_1^n - e^\tau (\mu_n \beta_1^n + A \chi_n \alpha_1) - e^{\frac{\tau}{2}} A \sum_{m=1}^{\infty} \chi_{n,m} \beta_0^m - D_{td} \gamma_n(0) e^{\frac{\tau}{2}} e^{-\mu_n(e^\tau - 1)} \\ \dot{\beta}_k^n &= -\frac{k}{2} \beta_k^n - e^\tau (\mu_n \beta_k^n + A \chi_n \alpha_k) - D_{td} \beta_{k-2}^n - e^{\frac{\tau}{2}} A \sum_{m=1}^{\infty} \chi_{n,m} \beta_{k-1}^m \quad 2 \leq k \leq N. \end{aligned} \quad (2.10)$$

Note that we have used the following facts. First,  $\langle H_k^{td}, w_0^s \rangle_{L^2} = 0$ , which follows by construction. This implies that  $\langle H_k^{td}, \mathcal{L}_{td} w_0^s \rangle_{L^2} = \langle -(k/2) H_k^{td}, w_0^s \rangle_{L^2} = 0$ . One can also check that

$$\mathcal{L}_{td}^* (\partial_\xi H_k^{td}) = -\frac{(k-1)}{2} \partial_\xi H_k^{td} \quad \Rightarrow \quad H_{k-1}^{td} = C_k \partial_\xi H_k^{td}$$

for some constant  $C_k$ , which implies that  $\langle H_k^{td}, \partial_\xi^2 w_0^s \rangle = 0$ . Similar results hold for  $v_n^s$ .

The key aspect of (2.10) is that, because of the structure of (2.7)-(2.8), the dynamics of  $\{\alpha_k\}_{k=0}^N$  and  $\{\beta_k^n\}_{k=0}^N$  do not depend on the remainder terms  $w_0^s$  or  $v_n^s$ . Therefore, the behavior of these low modes can be analyzed without any a priori knowledge of the remainder terms. The structure of the above system suggests that, with the exception of  $\alpha_0$ , everything should decay exponentially fast in  $\tau$ , which corresponds to algebraic decay in  $t$ . Moreover, the leading order behavior will be governed by  $\alpha_0$ .

### 2.3 Definition of $u_{\text{app}}$ and $u_{\text{rem}}$

We now relate the decomposition in (2.9) back to the solution  $u(X, y, z, T)$  of the original equation (1.3). We define  $u_{\text{app}}$  in terms of the low modes and  $u_{\text{rem}}$  in terms of the functions  $w_0^s$  and  $v_n^s$ . To do so we need to convert back to the  $(X, T)$  variables and take into account the decomposition in (1.4) and the change of variables in §2.1. In particular, we have

$$\begin{aligned} u(X, y, z, T) &= \sum_{n=0}^{\infty} u_n(X, T) \psi_n(y, z) \\ u_0(X, T) &= \frac{1}{\sqrt{1+T}} w_0(\xi, \tau) \\ u_n(X, T) &= \frac{1}{(T+1)} \left[ \gamma_n(\tau) \varphi_0^{td}(\xi) + \partial_{\xi} v_n(\xi, \tau) \right], \quad n \geq 1. \end{aligned}$$

Using (2.9), we find

$$\begin{aligned} u_0(X, T) &= \frac{1}{\sqrt{1+T}} \sum_{k=0}^N \alpha_k(\tau) \varphi_k^{td}(\xi) + \frac{1}{\sqrt{T+1}} w_0^s(\xi, \tau) \\ u_n(X, T) &= \frac{1}{(T+1)} \left[ \gamma_n(\tau) \varphi_0^{td}(\xi) + \sum_{k=0}^N \beta_k^n(\tau) \varphi_{k+1}^{td}(\xi) \right] + \frac{1}{(T+1)} \partial_{\xi} v_n^s(\xi, \tau). \end{aligned}$$

We now define

$$\begin{aligned} u_{\text{app}}(X, y, z, T) &= \frac{\psi_0(y, z)}{\sqrt{1+T}} \sum_{k=0}^N \alpha_k [\log(T+1)] \varphi_k^{td} \left( \frac{X}{\sqrt{T+1}} \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{\psi_n(y, z)}{(T+1)} \left[ \gamma_n [\log(T+1)] \varphi_0^{td} \left( \frac{X}{\sqrt{T+1}} \right) + \sum_{k=0}^N \beta_k^n [\log(T+1)] \varphi_{k+1}^{td} \left( \frac{X}{\sqrt{T+1}} \right) \right], \end{aligned} \quad (2.11)$$

and

$$u_{\text{rem}}(X, y, z, T) = \frac{\psi_0(y, z)}{\sqrt{1+T}} w_0^s \left( \frac{X}{\sqrt{T+1}}, \log(T+1) \right) + \sum_{k=1}^{\infty} \frac{\psi_k(y, z)}{(T+1)} \partial_{\xi} v_k^s \left( \frac{X}{\sqrt{T+1}}, \log(T+1) \right) \quad (2.12)$$

The behavior of  $u_{\text{app}}$ , as stated in Theorem 1(i), will be determined in §3, and the behavior of  $u_{\text{rem}}$ , as stated in Theorem 1(ii), will be determined in §4.

## 3 Taylor dispersion for the approximate solution via a center manifold

The main goal of this section is to prove Theorem 1(i). This will essentially be done via Proposition 3.2, and it will be explained in §3.2 how its proof follows from that Proposition.

### 3.1 Asymptotic behavior of the low modes via a center-stable manifold

Consider system (2.10). To construct its center manifold, we start by performing some changes of variables. Recall from the formal analysis that, in long time limit, we expect  $\mu_n w_n + A \chi_n \partial_{\xi} w_0 = 0$ . In system (2.10),

this results from the term  $e^\tau(\mu_n\beta_k^n + A\chi_n\alpha_k)$ . Therefore, we will diagonalize the system so that, in terms of new variables  $(a_k, b_k^n)$ , the set  $\{\mu_n\beta_k^n + A\chi_n\alpha_k\} = 0$  corresponds to the set  $\{b_k^n = 0\}$ . We define

$$a_k = \alpha_k, \quad b_k^n = \beta_k^n + \frac{A\chi_n}{\mu_n}\alpha_k \quad (3.1)$$

and obtain

$$\begin{aligned} \dot{a}_0 &= 0 \\ \dot{a}_1 &= -\frac{1}{2}a_1 - Ae^{\frac{\tau}{2}} \sum_{m=1}^{\infty} \chi_m \gamma_m(0) e^{-\mu_m(e^\tau-1)} \\ \dot{a}_k &= -\frac{k}{2}a_k - A \sum_{m=1}^{\infty} \chi_m b_{k-2}^m \quad 2 \leq k \leq N \\ \dot{b}_0^n &= -e^\tau \mu_n b_0^n - e^\tau A \sum_{m=1}^{\infty} \chi_{n,m} \gamma_m(0) e^{-\mu_m(e^\tau-1)} \\ \dot{b}_1^n &= -\left(\frac{1}{2} + e^\tau \mu_n\right) b_1^n - e^{\frac{\tau}{2}} A \sum_{m=1}^{\infty} \chi_{n,m} \left[ b_0^m - \frac{A\chi_m}{\mu_m} a_0 \right] - D_{td} \gamma_n(0) e^{\frac{\tau}{2}} e^{-\mu_n(e^\tau-1)} \\ &\quad - \frac{A^2 \chi_n}{\mu_n} e^{\frac{\tau}{2}} \sum_{m=1}^{\infty} \chi_m \gamma_m(0) e^{-\mu_m(e^\tau-1)} \\ \dot{b}_k^n &= -\left(\frac{k}{2} + e^\tau \mu_n\right) b_k^n - D_{td} \left( b_{k-2}^n - \frac{A\chi_n}{\mu_n} a_{k-2} \right) - \frac{A^2 \chi_n}{\mu_n} \sum_{m=1}^{\infty} \chi_m b_{k-2}^m \\ &\quad - e^{\tau/2} A \sum_{m=1}^{\infty} \chi_{n,m} \left( b_{k-1}^m - \frac{A\chi_m}{\mu_m} a_{k-1} \right) \quad 2 \leq k \leq N, \end{aligned}$$

where  $n \geq 1$ . This system is non-autonomous, which makes it difficult to construct a center manifold. To overcome this, we first undo the change of variables in time using  $\tau = \log(1+T)$  and define  $\sigma = (1+T)^{-1/2}$ . Denoting  $d/dT = (\cdot)'$ , we obtain

$$\begin{aligned} a'_0 &= 0 \\ a'_1 &= -\frac{1}{2}\sigma^2 a_1 - A\sigma \sum_{m=1}^{\infty} \chi_m \gamma_m(0) e^{-\mu_m T} \\ a'_k &= \sigma^2 \left( -\frac{k}{2} a_k - A \sum_{m=1}^{\infty} \chi_m b_{k-2}^m \right) \quad 2 \leq k \leq N \quad (3.2) \\ b'_0^n &= -\mu_n b_0^n - A \sum_{m=1}^{\infty} \chi_{n,m} \gamma_m(0) e^{-\mu_m T} \\ b'_1^n &= -\left(\frac{1}{2}\sigma^2 + \mu_n\right) b_1^n - A\sigma \sum_{m=1}^{\infty} \chi_{n,m} \left( b_0^m - \frac{A\chi_m}{\mu_m} a_0 \right) - D_{td} \sigma \gamma_n(0) e^{-\mu_n T} - \sigma \frac{A^2 \chi_n}{\mu_n} \sum_{m=1}^{\infty} \chi_m \gamma_m(0) e^{-\mu_m T} \\ b'_k^n &= -\left(\frac{k}{2}\sigma^2 + \mu_n\right) b_k^n - D_{td} \sigma^2 \left( b_{k-2}^n - \frac{A\chi_n}{\mu_n} a_{k-2} \right) - \frac{A^2 \chi_n}{\mu_n} \sigma^2 \sum_{m=1}^{\infty} \chi_m b_{k-2}^m \\ &\quad - \sigma A \sum_{m=1}^{\infty} \chi_{n,m} \left( b_{k-1}^m - \frac{A\chi_m}{\mu_m} a_{k-1} \right) \quad 2 \leq k \leq N \\ \sigma' &= -\frac{1}{2}\sigma^3, \end{aligned}$$

where  $n \geq 1$ . Note that, except for the terms involving  $\gamma_n(0)$ , which are decaying exponentially fast in  $T$ , this system is autonomous (but nonlinear), due to our definition of  $\sigma$ .

It is now convenient to define more compact notation. To that end, we write

$$b_k = (b_k^1, b_k^2, b_k^3, \dots), \quad \tilde{\chi} = (\chi_1, \chi_2, \chi_3, \dots), \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \dots),$$

where  $\tilde{\chi}$  is a constant,  $\gamma = \gamma(T)$  with  $\gamma_n(T) = \gamma_n(0)e^{-\mu_n T}$ , and  $b_k = b_k(T)$ , and  $n \geq 1$ . We also define operators on  $\ell^2$  via

$$(\tilde{\chi} * Y)_n = \sum_m \chi_{n,m} Y_m, \quad (\Upsilon Y)_n = \mu_n Y_n.$$

Throughout the following estimates we will use the following Lemma, which says that  $\tilde{\chi}$  and  $\Upsilon^{-1}$  are bounded operators.

**Lemma 3.1.** *The operators  $\tilde{\chi}$  and  $\Upsilon^{-1}$  are bounded operators on  $\ell^2$ .*

**Proof.** The bound on  $\Upsilon^{-1}$  follows immediately by noting that  $\|\Upsilon^{-1}Y\|_{\ell^2}^2 = \sum_{n=1}^{\infty} \mu_n^{-2} |Y_n|^2 \leq \mu_1^{-2} \|Y\|_{\ell^2}^2$  since  $\mu_n \geq \mu_1$  for all  $n \geq 1$ . The boundedness of  $\tilde{\chi}$  follows by noting that

$$(\tilde{\chi} * Y)_n = \sum_m \langle \psi_n, \chi \psi_m \rangle Y_m = \langle \psi_n, \chi \mathcal{Y} \rangle$$

where  $\mathcal{Y}(y, z) = \sum_m Y_m \psi_m(z, z)$ . Thus,  $(\tilde{\chi} * Y)_n$  is the generalized Fourier coefficient of the function  $\chi \mathcal{Y}$  and hence, by Parseval's equality.

$$\sum_n |(\tilde{\chi} * Y)_n|^2 = \|\chi \mathcal{Y}\|_{L^2(\Omega)}^2 \leq \|\chi\|_{L^\infty}^2 \|\mathcal{Y}\|_{L^2(\Omega)}^2 = \|\chi\|_{L^\infty}^2 \|Y\|_{\ell^2}^2, \quad (3.3)$$

where the last step in this expression again used Parseval's equality ■

With this notation, (3.2) can be written

$$\begin{aligned} a'_0 &= 0 \\ a'_1 &= -\frac{1}{2}\sigma^2 a_1 - A\sigma \langle \tilde{\chi}, \gamma \rangle_{\ell^2} \\ a'_k &= -\frac{k}{2}\sigma^2 a_k - A\sigma^2 \langle \tilde{\chi}, b_{k-2} \rangle_{\ell^2} \quad 2 \leq k \leq N \\ b'_0 &= -\Upsilon b_0 - A\tilde{\chi} * \gamma \\ b'_1 &= -\left(\frac{1}{2}\sigma^2 + \Upsilon\right) b_1 - A\sigma \tilde{\chi} * [b_0 - Aa_0(\Upsilon^{-1}\tilde{\chi})] - D_{td}\sigma\gamma - \sigma A^2 \langle \tilde{\chi}, \gamma \rangle (\Upsilon^{-1}\tilde{\chi}) \\ b'_k &= -\left(\frac{k}{2}\sigma^2 + \Upsilon\right) b_k - \sigma A\tilde{\chi} * [b_{k-1} - Aa_{k-1}(\Upsilon^{-1}\tilde{\chi})] - \sigma^2 D_{td} [b_{k-2} - Aa_{k-2}(\Upsilon^{-1}\tilde{\chi})] \\ &\quad - \sigma^2 A^2 \langle \tilde{\chi}, b_{k-2} \rangle (\Upsilon^{-1}\tilde{\chi}) \quad 2 \leq k \leq N \\ \sigma' &= -\frac{1}{2}\sigma^3 \end{aligned} \quad (3.4)$$

Intuitively, there are no linear terms in the equations for  $\{a_k\}_{k=0}^N$  (except for the term  $-A\sigma \langle \tilde{\chi}, \gamma \rangle$ , which is decaying exponentially fast) or in the equation for  $\sigma$ . The equations for  $\{b_k\}_{k=0}^N$  each contain a linear term of the form  $-\Upsilon b_k$ , where  $\langle \Upsilon b_k, b_k \rangle \geq \mu_1 \|b_k\|^2$ , with  $\mu_1 > 0$ . Hence, these variables should decay

exponentially fast, and there is a spectral gap determined by  $\mu_1$ . Therefore, there should exist an invariant center-stable manifold of dimension  $N + 2$  of the form  $\mathcal{M} = \{b_k = h_k(a_0, a_1, \dots, a_N, \sigma) : k = 0, 1, \dots, N\}$ . To see this, we note that  $\gamma' = -\Upsilon\gamma$  and add this equation to the above system to obtain the autonomous system

$$\begin{aligned}
a'_0 &= 0 \\
a'_1 &= -\frac{1}{2}\sigma^2 a_1 - A\sigma\langle\tilde{\chi}, \gamma\rangle_{\ell^2} \\
a'_k &= -\frac{k}{2}\sigma^2 a_k - A\sigma^2\langle\tilde{\chi}, b_{k-2}\rangle_{\ell^2} \quad 2 \leq k \leq N \\
b'_0 &= -\Upsilon b_0 - A\tilde{\chi} * \gamma \\
b'_1 &= -\left(\frac{1}{2}\sigma^2 + \Upsilon\right) b_1 - A\sigma\tilde{\chi} * [b_0 - Aa_0(\Upsilon^{-1}\tilde{\chi})] - D_{td}\sigma\gamma - \sigma A^2\langle\tilde{\chi}, \gamma\rangle(\Upsilon^{-1}\tilde{\chi}) \\
b'_k &= -\left(\frac{k}{2}\sigma^2 + \Upsilon\right) b_k - \sigma A\tilde{\chi} * [b_{k-1} - Aa_{k-1}(\Upsilon^{-1}\tilde{\chi})] - \sigma^2 D_{td} [b_{k-2} - Aa_{k-2}(\Upsilon^{-1}\tilde{\chi})] \\
&\quad - \sigma^2 A^2\langle\tilde{\chi}, b_{k-2}\rangle(\Upsilon^{-1}\tilde{\chi}) \quad 2 \leq k \leq N \\
\sigma' &= -\frac{1}{2}\sigma^3 \\
\gamma' &= -\Upsilon\gamma.
\end{aligned} \tag{3.5}$$

The linear part of this system (although no longer diagonal, due to the term  $-A\tilde{\chi} * \gamma$  in the  $b_0$  equation) now makes the spectral separation clear. One could abstractly justify the existence of a center manifold of the form  $(b_0, \dots, b_N, \gamma) = H(a_0, \dots, a_N, \sigma)$ . However, it turns out we can compute the function  $H$  explicitly, and it has a rather simple form. Moreover, we can show directly that the center manifold is globally attracting. These results are collected in the following proposition.

**Proposition 3.2.** *For each  $1 \leq k \leq N$ , there exist functions  $h_k = h_k(a_0, \dots, a_{k-1}, \sigma)$  of the form*

$$h_k(a_0, a_1, \dots, a_{k-1}, \sigma) = \sum_{\ell=1}^k C_{k-\ell}^k a_{k-\ell} \sigma^\ell, \tag{3.6}$$

where the  $C_{k-\ell}^k$  are elements of  $\ell^2$  for each  $k$  and  $\ell$ , can be computed explicitly, are independent of  $\nu$ , and such that (3.5) has an invariant center-stable manifold given by

$$\mathcal{M}_N = \{(b_0, \dots, b_N, \gamma) = (0, h_1(a_0, \sigma), \dots, h_N(a_0, \dots, a_{N-1}, \sigma), 0)\}. \tag{3.7}$$

Moreover, there exist constants  $C, \eta > 0$  that are independent of  $\nu$  and such that all solutions to (3.5) satisfy

$$\|(b_0, \dots, b_N, \gamma)(T) - (0, h_1(a_0, \sigma), \dots, h_N(a_0, \dots, a_{N-1}, \sigma), 0)\|_{(\ell^2)^{N+2}} \leq Ce^{-\eta T}, \tag{3.8}$$

where  $(a_0, \dots, a_{N-1})$  and  $\sigma$  are solutions of

$$\begin{aligned}
a'_0 &= 0 \\
a'_1 &= -\frac{1}{2}\sigma^2 a_1 \\
a'_k &= -\frac{k}{2}\sigma^2 a_k - A\sigma^2\langle\tilde{\chi}, h_{k-2}(a_0, \dots, a_{k-3}, \sigma)\rangle_{\ell^2} \quad 2 \leq k \leq N \\
\sigma' &= -\frac{1}{2}\sigma^3.
\end{aligned}$$

Moreover, for all  $k \geq 1$ ,

$$|a_k(\tau)| \leq Ce^{-\eta\tau}, \quad \tau = \log(1+T). \quad (3.9)$$

**Remark 3.3.** More precise statements of the convergence to the center manifold and decay within the center manifold are given in Lemmas 3.4 and 3.5, respectively. Note that the exponential in  $T$  convergence to the center manifold is equivalent to super-exponential in  $\tau$  convergence,  $e^{-\eta T} = e^{-\eta(e^\tau - 1)}$ , while the exponential in  $\tau$  decay on the center manifold, implied by (3.9), is equivalent to algebraic in  $T$  decay,  $e^{-\eta\tau} = (1+T)^{-\eta}$ . Furthermore, the  $\nu$ -independence of the constants  $C_{k-\ell}^k$  follows from the change of variable (1.2).

**Proof.** The Proof will be divided into three steps: 1) Justifying (3.6), the explicit formula for the center manifold; 2) Proving global convergence to the center manifold and justifying (3.8); and 3) Justifying equation (3.9), the decay rate within the center manifold.

**Step 1: Explicit formula for the center manifold** To justify (3.6), we will ultimately use induction, but we compute the first few terms directly since the equations in (3.5) are different for  $k = 0, 1$ . First, notice that the set  $(b_0, \gamma) = (0, 0)$  is invariant for (3.5). Next, we look for a function of the form

$$h_1(a_0, \sigma) = C_0^1 a_0 \sigma, \quad C_0^1 \in \ell^2,$$

so that the set  $(b_0, b_1, \gamma) = (0, h_1(a_0, \sigma), 0)$  is invariant. Computing  $(b_0, b_1, \gamma)'$  in two different ways and equating the results, we find that we need

$$-\frac{C_0^1}{2} a_0 \sigma^3 = -\frac{C_0^1}{2} a_0 \sigma^3 + \sigma a_0 [-\Upsilon C_0^1 + A^2 \tilde{\chi} * (\Upsilon^{-1} \tilde{\chi})].$$

Thus, we can take

$$C_0^1 = A^2 \Upsilon^{-1} \tilde{\chi} * (\Upsilon^{-1} \tilde{\chi}).$$

Next, we look for a function of the form

$$h_2(a_0, a_1, \sigma) = C_1^2 a_1 \sigma + C_0^2 a_0 \sigma^2$$

so that the set  $(b_0, b_1, b_2, \gamma) = (0, h_1(a_0, \sigma), h_2(a_0, a_1, \sigma), 0)$  is invariant. As above, we find

$$C_1^2 = A^2 \Upsilon^{-1} \tilde{\chi} * (\Upsilon^{-1} \tilde{\chi}), \quad C_0^2 = D_{td} A \Upsilon^{-2} \tilde{\chi} - A^3 \Upsilon^{-1} [\tilde{\chi} * (\tilde{\chi} * (\Upsilon^{-1} \tilde{\chi}))].$$

We now assume that (3.6) holds for  $0 \leq k \leq n$  and prove this implies it is true for  $k = n+1$  with  $n \geq 2$ .

First, we compute

$$\begin{aligned} b'_{n+1} &= \frac{d}{dt} \sum_{\ell=1}^{n+1} C_{n+1-\ell}^{n+1} a_{n+1-\ell} \sigma^\ell \\ &= \sum_{\ell=1}^{n+1} \left( -\frac{1}{2} \right) \ell C_{n+1-\ell}^{n+1} a_{n+1-\ell} \sigma^{\ell+2} - \frac{1}{2} C_1^{n+1} a_1 \sigma^{n+2} - \sigma^2 C_2^{n+1} a_2 \sigma^{n-1} \\ &\quad - \sum_{\ell=1}^{n-2} \frac{(n+1-\ell)}{2} C_{n+1-\ell}^{n+1} \sigma^{\ell+2} a_{n+1-\ell} - A \sum_{\ell=1}^{n-2} C_{n+1-\ell}^{n+1} \sigma^{\ell+2} \left\langle \tilde{\chi}, \sum_{j=1}^{n-\ell-1} C_{n-\ell-1-j}^{n-\ell-1} \sigma^j a_{n-\ell-1-j} \right\rangle \\ &= -\frac{(n+1)}{2} \sum_{\ell=1}^{n+1} C_{n+1-\ell}^{n+1} \sigma^{\ell+2} a_{n+1-\ell} - A \sum_{\ell=1}^{n-2} C_{n+1-\ell}^{n+1} \sigma^{\ell+2} \left\langle \tilde{\chi}, \sum_{j=1}^{n-\ell-1} C_{n-\ell-1-j}^{n-\ell-1} \sigma^j a_{n-\ell-1-j} \right\rangle. \end{aligned} \quad (3.10)$$

Using (3.5) and evaluating at  $b_k = h_k$ ,  $b_0 = \gamma = 0$ , we also have

$$\begin{aligned}
b'_{n+1} = & - \left[ \frac{(n+1)}{2} \sigma^2 + \Upsilon \right] \sum_{\ell=1}^{n+1} C_{n+1-\ell}^{n+1} a_{n+1-\ell} \sigma^\ell - A \tilde{\chi} * \sum_{\ell=1}^n C_{n-\ell}^n a_{n-\ell} \sigma^{\ell+1} \\
& - D_{td} \sum_{\ell=1}^{n-1} C_{n-1-\ell}^{n-1} a_{n-1-\ell} \sigma^{\ell+2} + \sigma A^2 a_n \tilde{\chi} * (\Upsilon^{-1} \check{\chi}) + D_{td} A \sigma^2 a_{n-1} (\Upsilon^{-1} \check{\chi}) \\
& - A^2 (\Upsilon^{-1} \check{\chi}) \left\langle \check{\chi}, \sum_{\ell=1}^{n-1} C_{n-1-\ell}^{n-1} a_{n-1-\ell} \sigma^{\ell+2} \right\rangle. \tag{3.11}
\end{aligned}$$

We now equate the expressions on the right hand sides of equations (3.10)-(3.11) to obtain

$$\begin{aligned}
& -A \sum_{\ell=1}^{n-2} C_{n+1-\ell}^{n+1} \sigma^{\ell+2} \left\langle \check{\chi}, \sum_{j=1}^{n-\ell-1} C_{n-\ell-1-j}^{n-\ell-1} \sigma^j a_{n-\ell-1-j} \right\rangle \\
& = -\Upsilon \sum_{\ell=1}^{n+1} C_{n+1-\ell}^{n+1} a_{n+1-\ell} \sigma^\ell - A \tilde{\chi} * \sum_{\ell=1}^n C_{n-\ell}^n a_{n-\ell} \sigma^{\ell+1} - D_{td} \sum_{\ell=1}^{n-1} C_{n-1-\ell}^{n-1} a_{n-1-\ell} \sigma^{\ell+2} \\
& \quad + \sigma A^2 a_n \tilde{\chi} * (\Upsilon^{-1} \check{\chi}) + D_{td} A \sigma^2 a_{n-1} (\Upsilon^{-1} \check{\chi}) - A^2 (\Upsilon^{-1} \check{\chi}) \left\langle \check{\chi}, \sum_{\ell=1}^{n-1} C_{n-1-\ell}^{n-1} a_{n-1-\ell} \sigma^{\ell+2} \right\rangle.
\end{aligned}$$

First, consider the resulting terms involving  $a_n$ . We need

$$0 = -\Upsilon C_n^{n+1} a_n \sigma + \sigma A^2 a_n \tilde{\chi} * (\Upsilon^{-1} \check{\chi}) \quad \Rightarrow \quad C_n^{n+1} = A^2 \Upsilon^{-1} \tilde{\chi} * (\Upsilon^{-1} \check{\chi}).$$

The terms involving  $a_{n-1}$  imply

$$0 = -\Upsilon C_{n-1}^{n+1} - A \tilde{\chi} * C_{n-1}^n + D_{td} A (\Upsilon^{-1} \check{\chi}) \quad \Rightarrow \quad C_{n-1}^{n+1} = D_{td} A \Upsilon^{-2} \check{\chi} - A \Upsilon^{-1} (\tilde{\chi} * C_{n-1}^n).$$

The terms involving  $a_{n-2}$  imply

$$\begin{aligned}
0 = & -\Upsilon C_{n-2}^{n+1} - A \tilde{\chi} * C_{n-2}^n - D_{td} C_{n-2}^{n-1} - A (\Upsilon^{-1} \check{\chi}) \langle \check{\chi}, C_{n-2}^{n-1} \rangle \\
\Rightarrow & C_{n-2}^{n+1} = \Upsilon^{-1} [-A \tilde{\chi} * C_{n-2}^n - D_{td} C_{n-2}^{n-1} - A (\Upsilon^{-1} \check{\chi}) \langle \check{\chi}, C_{n-2}^{n-1} \rangle].
\end{aligned}$$

Finally, for  $3 \leq k \leq n$ , the terms involving  $a_{n-k}$  imply

$$-A \sum_{\ell=1}^{n-2} C_{n+1-\ell}^{n+1} \langle \check{\chi}, C_{n-k}^{n-\ell-1} \rangle = -\Upsilon C_{n-k}^{n+1} - A \tilde{\chi} * C_{n-k}^n - D_{td} C_{n-k}^{n-1} - A^2 (\Upsilon^{-1} \check{\chi}) \langle \check{\chi}, C_{n-k}^{n-1} \rangle,$$

which gives

$$C_{n-k}^{n+1} = \Upsilon^{-1} \left[ A \sum_{\ell=1}^{n-2} C_{n+1-\ell}^{n+1} \langle \check{\chi}, C_{n-k}^{n-\ell-1} \rangle - A \tilde{\chi} * C_{n-k}^n - D_{td} C_{n-k}^{n-1} - A^2 (\Upsilon^{-1} \check{\chi}) \langle \check{\chi}, C_{n-k}^{n-1} \rangle \right].$$

All of the coefficients appearing in the sums on the RHS of this expression have been computed at previous stages of the iteration and hence we obtain  $C_{n-k}^{n+1}$  in the form asserted in the Proposition.

**Step 2: Proving global convergence to the center manifold and justifying (3.8):** We'll show that the exact invariant manifolds previously constructed are globally attracting. First, note that we can solve (3.5) explicitly to find

$$\gamma_n(T) = \gamma_n(0) e^{-\mu_n T} \quad \Rightarrow \quad \|\gamma(T)\|_{\ell^2} \leq e^{-\mu_1 T} \|\gamma(0)\|_{\ell^2}, \tag{3.12}$$



$$b_0(T) = e^{-\Upsilon T} b_0(0) - \int_0^T e^{-\Upsilon(T-s)} A \tilde{\chi} * \gamma(s) ds \quad \Rightarrow \quad \|b_0(T)\|_{\ell^2} \leq C(\|b_0(0)\|_{\ell^2}, \|\gamma(0)\|_{\ell^2})(1+T)e^{-\mu_1 T}, \quad (3.13)$$

and

$$\sigma(T) = \frac{1}{\sqrt{T+1}} \quad \Rightarrow \quad |\sigma(T)| \leq 1. \quad (3.14)$$

Next, define

$$B_k = b_k - h_k(a_0, \dots, a_{k-1}, \sigma), \quad k \geq 1, \quad (3.15)$$

where  $h_k$  is defined in (3.6).

**Lemma 3.4.** *There exists a  $C > 0$ , independent of  $\nu$ , such that for all  $t > 0$ ,*

$$\begin{aligned} \|B_1(T)\|_{\ell^2} &\leq C(1+T)^{\frac{3}{2}} e^{-\mu_1 T} \\ \|B_k(T)\|_{\ell^2} &\leq C(1+T)^{1+\frac{k}{2}} e^{-\mu_1 T} \quad 2 \leq k \leq N. \end{aligned}$$

**Proof.** For  $k = 1$ , we can compute  $B'_1$  and solve the resulting equation explicitly to find

$$\begin{aligned} B_1(T) &= e^{-\Upsilon T - \frac{1}{2} \log(T+1)} B_1(0) \\ &\quad - \int_0^T e^{-\Upsilon(T-s) - \frac{1}{2}(\log(T+1) - \log(s+1))} \left[ \frac{A}{\sqrt{1+s}} \tilde{\chi} * b_0(s) + \frac{D_{td}}{\sqrt{1+s}} \gamma(s) + \frac{A^2}{\sqrt{1+s}} (\Upsilon^{-1} \tilde{\chi}) \langle \tilde{\chi}, \gamma(s) \rangle \right] ds. \end{aligned}$$

As a result,

$$\|B_1(T)\|_{\ell^2} \leq C(\|B_1(0)\|_{\ell^2}, \|b_0(0)\|_{\ell^2}, \|\gamma(0)\|_{\ell^2})(1+T)^{3/2} e^{-\mu_1 T}. \quad (3.16)$$

Next, for  $k \geq 2$ , we have

$$B'_k = - \left( \frac{k^2}{2} \sigma^2 + \Upsilon \right) B_k - \sigma A \tilde{\chi} * B_{k-1} - \sigma^2 D_{td} B_{k-2} - \sigma^2 A^2 \langle \tilde{\chi}, B_{k-2} \rangle (\Upsilon^{-1} \tilde{\chi}),$$

and so, assuming the result is true for  $k-1$ ,

$$\begin{aligned} \|B_k(T)\|_{\ell^2} &\leq e^{-\mu_1 T} \|B_k(0)\|_{\ell^2} \\ &\quad - C(\|B_{k-1}(0)\|_{\ell^2}, \|B_{k-2}(0)\|_{\ell^2}, \|\gamma(0)\|_{\ell^2}) \int_0^T e^{-\mu_1 T} \left[ \frac{1}{\sqrt{1+s}} \frac{(1+s)^{1+\frac{k-1}{2}}}{1} + \frac{1}{(1+s)} \frac{(1+s)^{1+\frac{k-2}{2}}}{1} \right] ds, \end{aligned}$$

which implies the result. ■

### Step 3: Justifying equation (3.9), the decay rate within the center manifold

The goal of this section is to compute the decay rates of the  $a_k$  by considering the system (3.2) reduced to its center manifold, which is given by

$$\begin{aligned} a'_0 &= 0 \\ a'_1 &= -\frac{1}{2} \sigma^2 a_1 \\ a'_k &= -\frac{k}{2} \sigma^2 a_k - A \sigma^2 \langle \tilde{\chi}, h_{k-2}(a_0, \dots, a_{k-3}, \sigma) \rangle_{\ell^2} \quad 2 \leq k \leq N \\ \sigma' &= -\frac{1}{2} \sigma^3. \end{aligned}$$

Converting back to  $\tau = \log(1 + T)$ , this becomes

$$\begin{aligned}\dot{a}_0 &= 0 \\ \dot{a}_1 &= -\frac{1}{2}a_1 \\ \dot{a}_k &= -\frac{k}{2}a_k - A(\check{\chi}, h_{k-2}(a_0, \dots, a_{k-3}, e^{-\frac{\tau}{2}}))_{\ell^2} \quad 2 \leq k \leq N.\end{aligned}$$

Using the fact that  $h_0 = 0$ , we see immediately that

$$a_0(\tau) = a_0(0), \quad a_1(\tau) = a_1(0)e^{-\frac{1}{2}\tau}, \quad a_2(\tau) = a_2(0)e^{-\tau}. \quad (3.17)$$

**Lemma 3.5.** *There exists a  $C > 0$ , independent of  $\nu$ , such that if we write  $k = 3j + n$  with  $j \in \mathbb{N} \cup \{0\}$  and  $n \in \{0, 1, 2\}$  then, for all  $\tau \geq 0$ ,*

$$|a_k(\tau)| \leq Ce^{-\frac{(j+n)}{2}\tau}, \quad 0 \leq k \leq N.$$

**Proof.** Using the bound for  $h_1$  in (3.6), we find

$$|a_3(\tau)| \leq e^{-\frac{3}{2}\tau}|a_3(0)| + C \int_0^\tau e^{-\frac{3}{2}(\tau-s)}|a_0(s)|e^{-\frac{s}{2}}ds,$$

which implies

$$|a_3(\tau)| \leq Ce^{-\frac{1}{2}\tau}.$$

A similar calculation shows

$$|a_4(\tau)| \leq Ce^{-\tau}, \quad |a_5(\tau)| \leq Ce^{-\frac{3}{2}\tau}.$$

Consider now general  $k$ , and assume the result holds for  $a_m$  with  $m \leq k - 1$ . Using (3.6), we have

$$|a_k(\tau)| \leq |a_k(0)|e^{-\frac{k}{2}\tau} + \int_0^\tau e^{-\frac{k}{2}(\tau-s)} \left( \sum_{\ell=1}^{k-2} Ca_{k-2-\ell}e^{-\frac{\ell}{2}s} \right) ds.$$

Notice that

$$\sum_{\ell=1}^{k-2} a_{k-2-\ell}e^{-\frac{\ell}{2}s} = a_{k-3}e^{-\frac{1}{2}s} + a_{k-4}e^{-s} + a_{k-5}e^{-\frac{3}{2}s} + \dots + a_1e^{-\frac{(k-3)}{2}s} + a_0e^{-\frac{(k-2)}{2}s}.$$

Thus, if  $k = 3j + n$ , we find

$$\sum_{\ell=1}^{k-2} a_{k-2-\ell}e^{-\frac{\ell}{2}s} \sim e^{-\frac{(j+n)}{2}s} + e^{-\frac{(j+n+2)}{2}s} + \dots + e^{-\frac{(3j+n-2)}{2}s}.$$

Thus, we find

$$|a_k(\tau)| \leq |a_k(0)|e^{-\frac{k}{2}\tau} + \int_0^\tau e^{-\frac{k}{2}(\tau-s)}Ce^{-\frac{(j+n)}{2}s}ds \leq Ce^{-\frac{(j+n)}{2}\tau}$$

as claimed. ■

This concludes the proof of Proposition 3.2. ■

### 3.2 Proof of Theorem 1(i)

We now show how Theorem 1(i) follows from Proposition 3.2. Recall the definition of  $u_{\text{app}}$  in (2.11). The dynamics of  $u_{\text{app}}$  are governed by the behavior of  $\{\alpha_k\}_{k=0}^N$  and  $\{\beta_k^n\}_{k=0}^N$ , where  $n = 1, 2, \dots$ . Their dynamics are governed by (2.10), which is a system of ODEs on  $\mathbb{R}^N \times (\ell^2(\mathbb{R}))^N$ . Proposition 3.2 shows that, after converting to the variables  $a_k, b_k^n$ , this system has a finite-dimensional globally attracting center manifold given by (3.7), and the rate of convergence to that center manifold is exponential in  $T$ , as given in (3.8). Finally, recalling that  $\alpha_k = a_k$ ,  $\beta_k^n = b_k^n - (A\chi_n/\mu_n)\alpha_k$ , and that the only term among  $a_k, b_k^n$  that is not decaying in time is  $a_0$ , one obtains the leading behavior of (2.11). This justifies the statements in 1(i).

## 4 Decay of the remainder via spectral decomposition and hypocoercivity

The goal of this section is to prove Theorem 1(ii), which states that the remainder terms decay rapidly. To that end, insert the expansion (2.9) into (2.7)-(2.8) and project off the first  $N + 1$  eigenfunctions to obtain

$$\begin{aligned} \partial_\tau w_0^s &= \mathcal{L}_{td} w_0^s - D_{td} \left[ \alpha_{N-1} \varphi_{N+1}^{td} + \alpha_N \varphi_{N+2}^{td} + \partial_\xi^2 w_0^s \right] \\ &\quad - A \sum_{m=1}^{\infty} \chi_m \left[ \beta_{N-1}^m \varphi_{N+1}^{td} + \beta_N^m \varphi_{N+2}^{td} + \partial_\xi^2 v_m^s \right] \end{aligned} \quad (4.1)$$

$$\begin{aligned} \partial_\tau v_n^s &= \mathcal{L}_{td} v_n^s - D_{td} \left[ \beta_{N-1}^n \varphi_{N+1}^{td} + \beta_N^n \varphi_{N+2}^{td} + \partial_\xi^2 v_n^s \right] - e^{\frac{\tau}{2}} A \sum_{m=1}^{\infty} \chi_{n,m} \left[ \beta_N^m \varphi_{N+1}^{td} + \partial_\xi v_m^s \right] \\ &\quad - e^\tau [\mu_n v_n^s + A \chi_m w_0^s]. \end{aligned}$$

The operator  $\mathcal{L}_{td}$ , acting on  $w_0^s$  and  $v_n^s$ , decays like  $e^{-\frac{N+1}{2}\tau}$ . In addition, the forcing terms in the above equation decay like  $\alpha_k, \beta_k$  with  $k \geq N - 1$ , which, due to Lemmas 3.4 - 3.5, decay like  $e^{-(j+n)\tau/2} \leq e^{-k\tau/6}$ , for  $k = 3j + n$ . Therefore, we expect  $w_0^s$  and  $v_n^s$  to decay with the same rate as the forcing terms.

To prove this, we will not work with the above system in the  $(\xi, \tau)$  variables, but we will instead work in the Fourier space associated with the original  $(X, T)$  variables. Using the fact that

$$u_0^s(X, T) = \frac{1}{\sqrt{T+1}} w_0^s \left( \frac{X}{\sqrt{T+1}}, \log(T+1) \right), \quad u_n^s(X, T) = \frac{1}{(T+1)} \partial_\xi v_n^s \left( \frac{X}{\sqrt{T+1}}, \log(T+1) \right), \quad (4.2)$$

we find

$$\begin{aligned}
\partial_T u_0^s &= \nu^2 \partial_X^2 u_0^s - A \sum_{m=1}^{\infty} \chi_m \partial_X u_m^s \\
&\quad - \frac{D_{td}}{(1+T)^{3/2}} \left[ \alpha_{N-1}(\log(T+1)) \varphi_{N+1}^{td} \left( \frac{X}{\sqrt{T+1}} \right) + \alpha_N(\log(T+1)) \varphi_{N+2}^{td} \left( \frac{X}{\sqrt{T+1}} \right) \right] \\
&\quad - \frac{A}{(1+T)^{3/2}} \sum_{m=1}^{\infty} \chi_m \left[ \beta_{N-1}^m(\log(T+1)) \varphi_{N+1}^{td} \left( \frac{X}{\sqrt{T+1}} \right) + \beta_N^m(\log(T+1)) \varphi_{N+2}^{td} \left( \frac{X}{\sqrt{T+1}} \right) \right] \\
\partial_T u_n^s &= \nu^2 \partial_X^2 u_n^s - A \sum_{m=1}^{\infty} \chi_{n,m} \partial_X u_m^s - [\mu_n u_n^s + A \chi_n \partial_X u_0^s] \\
&\quad - \frac{D_{td}}{(1+T)^2} \left[ \beta_{N-1}^n(\log(T+1)) \varphi_{N+2}^{td} \left( \frac{X}{\sqrt{T+1}} \right) + \beta_N^n(\log(T+1)) \varphi_{N+3}^{td} \left( \frac{X}{\sqrt{T+1}} \right) \right] \\
&\quad - \frac{A}{(1+T)^{3/2}} \sum_{m=1}^{\infty} \chi_{n,m} \beta_N^m(\log(T+1)) \varphi_{N+2}^{td} \left( \frac{X}{\sqrt{T+1}} \right).
\end{aligned}$$

If we now take the Fourier transform and use the notation

$$\hat{U}(\kappa, T) = \begin{pmatrix} \hat{u}_0^s(\kappa, T) \\ \{\hat{u}_n^s(\kappa, T)\}_{n=1}^{\infty} \end{pmatrix}, \quad \check{\chi} = \{\chi_n\}_{n=1}^{\infty}, \quad (\check{\chi} * f)_n = \sum_{m=1}^{\infty} \chi_{n,m} f_m, \quad (4.3)$$

we find

$$\frac{d}{dT} \hat{U} = \mathcal{B}(\kappa) \hat{U} + \hat{F}(\kappa, T), \quad (4.4)$$

where

$$\mathcal{B}(\kappa) = -\nu^2 \kappa^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\kappa A \begin{pmatrix} 0 & \check{\chi} \\ \check{\chi} & \check{\chi}^* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \Upsilon \end{pmatrix} =: \kappa^2 \mathcal{B}_2 + \kappa \mathcal{B}_1 + \mathcal{B}_0 \quad (4.5)$$

and

$$\hat{F}(\kappa, T) = \begin{pmatrix} \hat{F}_1(\kappa, T) \\ \hat{F}_2(\kappa, T) \end{pmatrix}, \quad (4.6)$$

with

$$\begin{aligned}
\hat{F}_1(\kappa, T) &= -\frac{D_{td} \hat{\Phi}_0^{td}(\kappa, T)}{(1+T)^{3/2}} \left[ \alpha_{N-1}(T) (1+T)^{\frac{N+1}{2}} (-i\kappa)^{N+1} + \alpha_N(T) (1+T)^{\frac{N+2}{2}} (-i\kappa)^{N+2} \right] \\
&\quad - \frac{A \hat{\Phi}_0^{td}(\kappa, T)}{(1+T)^{3/2}} \sum_{m=1}^{\infty} \chi_m \left[ \beta_{N-1}^m(T) (1+T)^{\frac{N+1}{2}} (-i\kappa)^{N+1} + \beta_N^m(T) (1+T)^{\frac{N+2}{2}} (-i\kappa)^{N+2} \right] \\
\hat{F}_2(\kappa, T) &= -\frac{D_{td} \hat{\Phi}_0^{td}(\kappa, T)}{(1+T)^2} \left[ \beta_{N-1}^n(T) (1+T)^{\frac{N+2}{2}} (-i\kappa)^{N+2} + \beta_N^n(T) (1+T)^{\frac{N+3}{2}} (-i\kappa)^{N+3} \right] \\
&\quad - \frac{A \hat{\Phi}_0^{td}(\kappa, T)}{(1+T)^{3/2}} \sum_{m=1}^{\infty} \chi_{n,m} \beta_N^m(T) (1+T)^{\frac{N+2}{2}} (-i\kappa)^{N+2}. \quad (4.7)
\end{aligned}$$

Note that we have written  $\alpha_j(\log(T+1)) = \alpha_j(T)$  and  $\beta_j^n(\log(T+1)) = \beta_j^n(T)$  for convenience, and a direct calculation shows that

$$\hat{\Phi}_0^{td}(\kappa, T) = \sqrt{T+1} e^{-\nu_{td} \kappa^2 (T+1)}.$$

The plan is to analyze the behavior of (4.4) using Duhamel's formula,

$$\hat{U}(\kappa, T) = e^{\mathcal{B}(\kappa)T} \hat{U}(\kappa, 0) + \int_0^T e^{\mathcal{B}(\kappa)(T-s)} \hat{F}(\kappa, s) ds, \quad (4.8)$$

and show that solutions decay like  $T^{-\mathcal{N}(N)}$ , where  $\mathcal{N}$  can be made large by choosing  $N$  large. The precise relationship between  $\mathcal{N}$  and  $N$  is given in the statement of Proposition 4.2. We will obtain this decay in the norm

$$\|\hat{U}(\cdot, T)\|^2 = \int_{\mathbb{R}} \|\hat{U}(\kappa, T)\|_Y^2 d\kappa = \int_{\mathbb{R}} |\hat{u}_0^s(\kappa, T)|^2 d\kappa + \int_{\mathbb{R}} \|\{\hat{u}_n^s(\kappa, T)\}\|_{\ell^2}^2 d\kappa. \quad (4.9)$$

**Remark 4.1.** Recall that we expect decay of the remainder terms  $w_0^s, v_n^s$  in  $L^2(m)$ , and the relationship between these variables and  $u_0, u_n$  is given in (4.2). Suppose that two functions  $g$  and  $f$  are related via

$$g(\xi, \tau) = (1 + T)^\gamma f(X, T), \quad \xi = \frac{X}{\sqrt{1 + T}}, \quad \tau = \log(1 + T).$$

Then we have

$$\begin{aligned} \|g(\tau)\|_{L^2(m)}^2 &= \int (1 + \xi^2)^m |g(\xi, \tau)|^2 d\xi = (1 + T)^{2\gamma-1/2} \int [1 + X^2(1 + T)^{-1}]^m |f(X, T)|^2 dX \\ &\simeq (1 + T)^{2\gamma-1/2} \sum_{j=0}^m (1 + T)^{-j} \int |X^j f(X, T)|^2 dX \\ &= (1 + T)^{2\gamma-1/2} \sum_{j=0}^m (1 + T)^{-j} \|\partial_\kappa^j \hat{f}(T)\|_{L^2}^2. \end{aligned}$$

The discussion at the beginning of this section suggests we can expect  $w_0^s(\xi, \tau)$  and  $v_n^s(\xi, \tau)$  to decay like

$$\|w_0^s(\tau)\|_{L^2(m)} + \|v_n^s(\tau)\|_{\ell^2} \|L^2(m) \sim e^{-\eta(N)\tau},$$

where  $\eta(N)$  grows with  $N$ . Therefore, one could estimate solutions to (4.8) in terms of the norm

$$\|\|\hat{U}(T)\|\| = (1 + T)^{1/2} \sum_{j=0}^m (1 + T)^{-j} \|\partial_\kappa^j \hat{u}_0(T)\|_{L^2}^2 + (1 + T)^{3/2} \sum_{j=0}^m (1 + T)^{-j} \|\partial_\kappa^j(\kappa^{-1} \|\{\hat{u}_n(T)\}\|_{\ell^2})\|_{L^2}^2. \quad (4.10)$$

Although this is possible [Cha17], the calculations are cumbersome. Therefore, we have chosen to carry out the estimates in terms of the much simpler norm (4.9), which also seems quite natural.

The goal of this section will be to prove the following result.

**Proposition 4.2.** For any  $N \in \mathbb{N}$  and  $\hat{U}(\kappa, 0)$  such that  $\|\partial_\kappa^\ell \hat{U}(\cdot, 0)\| < \infty$  for all  $0 \leq \ell \leq N + 1$ , the corresponding solution of (4.8) satisfies

$$\|\hat{U}(\cdot, T)\| \leq C(1 + T)^{-\frac{N}{6} - \frac{1}{12}}.$$

for all  $T \geq 0$ , where  $C$  is a constant that is independent of  $\nu$  but depends on  $\hat{U}(0)$  and its derivatives.

**Remark 4.3.** Note that the result claimed in Theorem 1(ii) follows from the above proposition. To see this, recall that  $u_{\text{rem}}$  is defined in (2.12). Using equations (4.2), (4.9), and Plancherel's Theorem, we have

$$\begin{aligned} \|u_{\text{rem}}(T)\|_{L^2}^2 &\leq C_\psi [\|u_0^s(T)\|_{L^2}^2 + \|\|\{u_n^s(T)\}\|_{\ell^2}^2\|_{L^2}^2] \\ &= C_\psi [\|\hat{u}_0^s(T)\|_{L^2}^2 + \|\|\{\hat{u}_n^s(T)\}\|_{\ell^2}^2\|_{L^2}^2] \\ &= C_\psi \|\hat{U}(T)\|^2 \leq C(1 + T)^{-\frac{N}{3} - \frac{1}{6}}, \end{aligned}$$

where  $C_\psi$  is a constant that depends on the  $L^2$  norms of the cross-sectional eigenfunctions  $\psi_n$ . Note that the requirement that  $\|\partial_\kappa^\ell \hat{U}(\cdot, 0)\| < \infty$  for all  $0 \leq \ell \leq N + 1$  in the above proposition holds as long as the initial data for (1.1) lies in the algebraically weighted function space:  $u(\cdot, 0) \in L^2(N + 1) \times L^2(\Omega)$ . This is because  $\partial_\kappa^\ell \hat{f} \in L^2$  if and only if  $X^\ell f \in L^2$ , which means  $f \in L^2(\ell)$ .

We now state a brief result on the decay of the forcing terms in (4.8).

**Lemma 4.4.** *There exists a constant  $C > 0$ , independent of  $\nu$ , such that, for all  $T > 0$ ,  $\kappa \in \mathbb{R}$*

$$\begin{aligned} |\hat{F}_1(\kappa, T)| &\leq C(1+T)^{\frac{N-1}{2}-\frac{1}{2}(j+n)} |\kappa|^{N+1} e^{-\nu_{td}\kappa^2(1+T)} [1 + |\kappa|(1+T)^{1/2}] \\ \|\hat{F}_2(\kappa, T)\|_{\ell^2} &\leq C(1+T)^{\frac{N-1}{2}-\frac{1}{2}(j+n)} |\kappa|^{N+2} e^{-\nu_{td}\kappa^2(1+T)} [1 + |\kappa|(1+T)^{1/2} + (1+T)^{1/2}] \end{aligned}$$

where  $n, j$  are defined so that  $N - 1 = 3j + n$ , with  $n \in \{0, 1, 2\}$ .

**Proof.** This is a direct consequence of the definition of  $\hat{F}$  in (4.7), of Lemmas 3.4 - 3.5, and of (3.1). ■

In order to combine Lemma 4.4 with equation (4.8) and prove a decay result for the remainder terms, we will need good control of the semigroup generated by  $\mathcal{B}(\kappa)$ . To obtain this, we will first obtain estimates on the spectrum of  $\mathcal{B}(\kappa)$ . We will then use these spectral estimates to obtain decay estimates on the semigroup for three different regions: 1) small wavenumber  $0 \leq |\kappa| \leq \kappa_0$ ; 2) intermediate wavenumber  $\kappa_0 \leq |\kappa| \leq \kappa_1 \nu^{-1}$ ; and 3) large wavenumber  $\kappa_1 \nu^{-1} \leq |\kappa|$ , where  $\kappa_0$  and  $\kappa_1$  are positive constants that are independent of  $\nu$ .

## 4.1 Spectral decomposition

First, we state a lemma on the spectrum of  $\mathcal{B}_{0,1,2}$ .

**Lemma 4.5.** *On the space  $Y = \mathbb{C} \times \ell^2(\mathbb{C})$  the following hold.*

- (i) *The operator  $\mathcal{B}_0$  has only point spectrum, and it is given by  $\sigma(\mathcal{B}_0) = \{0\} \cup \{-\mu_n\}_{n=1}^\infty$ .*
- (ii) *The operators  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are bounded.*

**Proof.**

- (i) This follows from the fact that  $\mathcal{B}_0$  is diagonal and the only accumulation point of its entries is  $\infty$ .
- (ii) This is trivially true for  $\mathcal{B}_2$  because it is a scalar multiple of the identity, and for  $\mathcal{B}_1$  it follows from the fact that  $\{\psi_n\}_{n=0}^\infty$  forms an orthonormal basis for  $L^2(\Omega)$  and Parseval's identity. ■

Next, we analyze the spectrum of  $\mathcal{B}(\kappa)$  for any fixed  $\kappa \in \mathbb{R}$ .

**Lemma 4.6.** *Fix any  $\kappa \in \mathbb{R}$ . The spectrum of  $\mathcal{B}(\kappa)$  consists only of point spectrum.*

**Proof.** We will show that, for fixed  $\kappa$ ,  $\mathcal{B}(\kappa) = \mathcal{B}_0 + \kappa(\mathcal{B}_1 + \kappa\mathcal{B}_2)$  is a relatively compact perturbation of  $\mathcal{B}_0$ . The result will then follow from Weyl's theorem [RS78, XIII.4, Corollary 2]. We must show

$$\kappa(\mathcal{B}_1 + \kappa\mathcal{B}_2)(\mathcal{B}_0 + i)^{-1}$$

is a compact operator on  $\mathbb{C} \times \ell^2(\mathbb{C})$ . By Parseval's identity, this is equivalent to showing that

$$\kappa(iA\chi(y, z) - \nu^2\kappa)(\Delta + i)^{-1}$$

is a compact operator on  $L^2(\Omega)$ . We let  $\{\hat{u}_n(y, z)\} \subset L^2(\Omega)$  be a bounded sequence:  $\|\hat{u}_n(y, z)\|_{L^2(\Omega)} \leq C$  for all  $n \in \mathbb{N}$ . Then, since  $i$  is in the resolvent set of  $\Delta$  and  $(\Delta + i)^{-1} : L^2(\Omega) \rightarrow H^1(\Omega)$  is bounded, it follows that  $\{(\Delta + i)^{-1}\hat{u}_n\}$  is a bounded sequence in  $H^1(\Omega)$ . Therefore

$$\{\kappa(iA\chi(y, z) - \nu^2\kappa)(\Delta + i)^{-1}\hat{u}_n\}$$

is also a bounded sequence in  $H^1(\Omega)$ . Since  $H^1(\Omega)$  is compactly embedded in  $L^2(\Omega)$ , this sequence has an  $L^2(\Omega)$  convergent subsequence. Therefore  $\kappa(iA\chi(y, z) - \nu^2\kappa)(\Delta + i)^{-1}$  is compact.  $\blacksquare$

#### 4.1.1 Low wavenumber estimates using the leading eigenvalue

We next prove a result on the spectrum of  $\mathcal{B}(\kappa)$  for  $|\kappa|$  sufficiently small. In particular, we show in this case that the eigenvalues of  $\mathcal{B}(\kappa)$  split into two parts: an eigenvalue  $\lambda_0(\kappa)$  near 0, and eigenvalues  $\lambda(\kappa)$  satisfying  $\text{Re}(\lambda(\kappa)) \leq -\mu_1/2$ . Therefore, we expect  $\lambda_0(\kappa)$  to dominate the long-time behavior, and we will therefore be able to use it to obtain estimates on the low-wavenumber part of our solution. In addition, we will show that this leading eigenvalue  $\lambda_0(\kappa)$  is approximately  $-\nu_{td}\kappa^2$ , so the long-time behavior will correspond with Taylor dispersion.

**Proposition 4.7.** *Let  $\kappa_0 = \min \left\{ \frac{\mu_1}{2}, \frac{1}{2A\|\chi\|_{L^\infty}} \left( \frac{1}{1 + \frac{\mu_1^2}{2}} \right) \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}} \right\}$ . Fix any  $\kappa \in \mathbb{R}$  such that  $|\kappa| \leq \kappa_0$ , and let  $0 < \nu < 1$ .*

- (i) *The (point) spectrum of  $\mathcal{B}(\kappa)$  can be divided into two disjoint sets,  $\sigma(\mathcal{B}(\kappa)) = \{\lambda_0(\kappa)\} \cup \Sigma(\kappa)$ , where  $|\lambda_0(\kappa)| \leq \sqrt{2}\mu_1/2$  and, for any eigenvalue  $\lambda(\kappa) \in \Sigma(\kappa)$  we have  $\text{Re}(\lambda(\kappa)) \leq -\mu_1/2$ .*
- (ii) *The leading eigenvalue satisfies  $\lambda_0(\kappa) = -\nu_{td}\kappa^2 + \Lambda_0(\kappa)$ , where  $\Lambda_0(\kappa) = ir\kappa^3 + \mathcal{O}(\kappa^4)$  is smooth, and independent of  $\nu$ . Here  $r = r(\chi, \{\mu_n\}_{n=1}^\infty) \in \mathbb{R}$  is given in equation (4.21).*

The main idea behind this Proposition is the following: recall that  $\mathcal{B}(\kappa) = \mathcal{B}_0 + \kappa\mathcal{B}_1 + \kappa^2\mathcal{B}_2$ . If  $|\kappa|$  is small, then  $\mathcal{B}(\kappa)$  is just a small perturbation of  $\mathcal{B}_0$ , which has spectrum  $\{0\} \cup \{-\mu_n\}_{n=1}^\infty$  and the separation claimed in (i). Furthermore, we will see that  $\mathcal{B}_1$  is antisymmetric, hence the real part of the spectrum of  $\mathcal{B}(\kappa)$  is actually an  $\mathcal{O}(\kappa^2)$  perturbation of that of  $\mathcal{B}_0$ . The  $\nu$ -dependence of the spectrum stated in the proposition can be obtained from the following decomposition: recall that  $\mathcal{B}_2 = -\nu^2\kappa^2I$ . Letting  $\mathcal{C}(\kappa) = \mathcal{B}_0 + \kappa\mathcal{B}_1$ , we have that  $\mathcal{B}(\kappa) = \mathcal{C}(\kappa) - \nu^2\kappa^2I$ . That is, the operators  $\mathcal{B}(\kappa)$  and  $\mathcal{C}(\kappa)$  differ by a scalar multiple of the identity, and, since  $\mathcal{C}(\kappa)$  is independent of  $\nu$ , all of the  $\nu$ -dependence of  $\mathcal{B}(\kappa)$  is contained in this scalar. Therefore we immediately have the following lemma:

**Lemma 4.8.** *Fix any  $\kappa \in \mathbb{R}$ , let  $\nu > 0$ , and let  $\mathcal{B}(\kappa)$  and  $\mathcal{C}(\kappa)$  be defined as above. The following are true:*

- (i) *The semigroups of  $\mathcal{B}(\kappa)$  and  $\mathcal{C}(\kappa)$  are related by  $e^{\mathcal{B}(\kappa)T} = e^{-\nu^2\kappa^2T} e^{\mathcal{C}(\kappa)T}$ .*
- (ii) *The eigenvalues  $\lambda(\kappa)$  of  $\mathcal{B}(\kappa)$  and  $\Gamma(\kappa)$  of  $\mathcal{C}(\kappa)$  are in one-to-one correspondence with one another via  $\lambda(\kappa) = \Gamma(\kappa) - \nu^2\kappa^2$ , and corresponding eigenvalues have the same projection operators  $P(\kappa)$ .*

**Remark 4.9.** Since the operator  $\mathcal{C}(\kappa)$  is independent of  $\nu$ , the above lemma tells us exactly what the  $\nu$ -dependence is in the semigroup  $e^{\mathcal{B}(\kappa)T}$ , and it tells us exactly what the  $\nu$ -dependence is in the eigenvalues  $\lambda(\kappa)$  in terms of the ( $\nu$ -independent) eigenvalues  $\Gamma(\kappa)$  of  $\mathcal{C}(\kappa)$ . Furthermore, since the projections  $P(\kappa)$  of corresponding eigenvalues are the same, and  $\mathcal{C}(\kappa)$  is independent of  $\nu$ , these projections can be taken to be independent of  $\nu$ . This relationship between the  $\nu$ -dependence and the structure of the system is a direct consequence of the change of variable (1.2).

Note that, because  $\mathcal{B}(\kappa)$  generates an analytic semigroup, the following Corollary follows immediately from Proposition 4.7(i).

**Corollary 4.10.** Let  $\kappa_0 = \min \left\{ \frac{\mu_1}{2}, \frac{1}{2A\|\chi\|_{L^\infty_\Omega}} \left( \frac{1}{1+\frac{\mu_1^2}{2}} \right) \frac{1}{\sqrt{1+(\frac{2}{\mu_1})^2}} \right\}$ . Fix any  $\kappa \in \mathbb{R}$  such that  $|\kappa| \leq \kappa_0$ , and let  $0 < \nu < 1$ . Let  $Q_0(\kappa)$  be the projection complementary to the eigenspace of the eigenvalue  $\lambda_0(\kappa)$  of  $\mathcal{B}(\kappa)$ . Then, for all  $W \in \mathbb{C} \times \ell^2(\mathbb{C}) = Y$  and  $T > 0$ , we have

$$\|e^{\mathcal{B}(\kappa)T} Q_0(\kappa)W\|_Y \leq Ce^{-\frac{\mu_1}{2}T} \|W\|_Y,$$

for some constant  $C > 0$  which is independent of  $\nu$ .

Before proving Proposition 4.7, we will need to prove the following Lemma.

**Lemma 4.11.** Let  $\kappa \in \mathbb{R}$ , let  $\kappa_0 = \min \left\{ \frac{\mu_1}{2}, \frac{1}{2A\|\chi\|_{L^\infty_\Omega}} \left( \frac{1}{1+\frac{\mu_1^2}{2}} \right) \frac{1}{\sqrt{1+(\frac{2}{\mu_1})^2}} \right\}$ , and let  $\lambda(\kappa)$  be an eigenvalue of  $\mathcal{B}(\kappa)$ . Then

(i)  $Re(\lambda(\kappa)) \leq -\nu^2\kappa^2$ .

(ii) If  $|\kappa| \leq \kappa_0$ , then  $|Im\lambda(\kappa)| < \mu_1/2$ .

**Proof.** Recall from (4.5) that  $\mathcal{B}(\kappa) = \mathcal{B}_0 + \kappa\mathcal{B}_1 + \kappa^2\mathcal{B}_2$  with

$$\mathcal{B}_0 = \begin{pmatrix} 0 & 0 \\ 0 & -\Upsilon \end{pmatrix}, \quad \mathcal{B}_1 = Ai \begin{pmatrix} 0 & \check{\chi} \\ \check{\chi} & \check{\chi}^* \end{pmatrix}, \quad \mathcal{B}_2 = -\nu^2 \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}.$$

Note that  $\mathcal{B}_0$  and  $\mathcal{B}_2$  are diagonal and hence

$$\mathcal{S}(\kappa) := \mathcal{B}_0 + \kappa^2\mathcal{B}_2$$

is symmetric. Also note that

$$\mathcal{A}(\kappa) := \kappa\mathcal{B}_1$$

is anti-symmetric, due to the following argument. Let  $V = \{V_n\}_{n=0}^\infty \in \mathbb{C} \times \ell^2(\mathbb{C})$  and let  $v(y, z) = V_0 + \sum_{n=1}^\infty V_n \psi_n(y, z)$ . We compute, using Parseval's identity,

$$\begin{aligned} \langle \mathcal{B}_1 V, V \rangle_{\mathbb{C} \times \ell^2(\mathbb{C})} &= \langle Ai\chi(y, z)v(y, z), v(y, z) \rangle_{L^2(\Omega)} \\ &= -\langle v(y, z), Ai\chi(y, z)v(y, z) \rangle_{L^2(\Omega)} \\ &= -\langle V, \mathcal{B}_1 V \rangle_{\mathbb{C} \times \ell^2(\mathbb{C})}. \end{aligned}$$



Next, we compute a bound on the spectrum of  $\mathcal{B}(\kappa) = \mathcal{S}(\kappa) + \mathcal{A}(\kappa)$ . Suppose  $\lambda(\kappa)$  is an eigenvalue of  $\mathcal{B}(\kappa)$  with eigenvector  $V(\kappa)$  normalized so that  $\|V(\kappa)\|_Y = 1$ . We have

$$\mathcal{B}(\kappa)V(\kappa) = \lambda(\kappa)V(\kappa).$$

Taking the  $\mathbb{C} \times \ell^2(\mathbb{C})$  inner product of both sides with  $V(\kappa)$ , we get

$$\langle \mathcal{B}(\kappa)V(\kappa), V(\kappa) \rangle = \langle \lambda(\kappa)V(\kappa), V(\kappa) \rangle = \lambda(\kappa)\langle V(\kappa), V(\kappa) \rangle = \lambda(\kappa).$$

Since  $\mathcal{B} = \mathcal{S} + \mathcal{A}$ , this implies

$$\begin{aligned} \langle (\mathcal{S}(\kappa) + \mathcal{A}(\kappa))V(\kappa), V(\kappa) \rangle &= \lambda(\kappa) \\ \langle \mathcal{S}(\kappa)V(\kappa), V(\kappa) \rangle + \langle \mathcal{A}(\kappa)V(\kappa), V(\kappa) \rangle &= \lambda(\kappa) \\ \langle V(\kappa), \mathcal{S}(\kappa)V(\kappa) \rangle - \langle V(\kappa), \mathcal{A}(\kappa)V(\kappa) \rangle &= \lambda(\kappa), \end{aligned} \tag{4.11}$$

where the last line is obtained using the symmetry properties of  $\mathcal{S}(\kappa)$  and  $\mathcal{A}(\kappa)$ . Taking the middle line in (4.11) and taking the complex conjugate, we have

$$\langle V(\kappa), \mathcal{S}(\kappa)V(\kappa) \rangle + \langle V(\kappa), \mathcal{A}(\kappa)V(\kappa) \rangle = \bar{\lambda}(\kappa). \tag{4.12}$$

Finally, adding (4.12) and the last line in (4.11) yields

$$Re(\lambda(\kappa)) = \langle V(\kappa), \mathcal{S}(\kappa)V(\kappa) \rangle_{\mathbb{C} \times \ell^2(\mathbb{C})}.$$

Next, we let  $v(y, z, \kappa) = V_0(\kappa) + \sum_{n=1}^{\infty} V_n(\kappa)\psi_n(y, z)$ , normalized so that  $\langle v(\kappa), v(\kappa) \rangle_{L^2(\Omega)} = 1$ , and apply Parseval's identity to get

$$\begin{aligned} Re(\lambda(\kappa)) &= \langle V(\kappa), \mathcal{S}(\kappa)V(\kappa) \rangle_{\mathbb{C} \times \ell^2(\mathbb{C})} \\ &= \int_{\Omega} v(y, z, \kappa)(\Delta - \nu^2\kappa^2)\bar{v}(y, z, \kappa)dydz \\ &= - \int_{\Omega} \nabla v(y, z, \kappa) \cdot \nabla \bar{v}(y, z, \kappa)dydz - \nu^2\kappa^2 \int_{\Omega} |v(y, z, \kappa)|^2 dydz \\ &\leq -\nu^2\kappa^2 \int_{\Omega} |v(y, z, \kappa)|^2 dydz \\ &= -\nu^2\kappa^2, \end{aligned} \tag{4.13}$$

which proves the first part of Lemma 4.11. For the second part of this lemma, let  $|\kappa| \leq \kappa_0$ . We use an argument similar to that used in the proof of the first part of this lemma to control the imaginary part of  $\lambda(\kappa)$ . Subtracting (4.12) and the last line in (4.11) yields an expression for  $Im(\lambda(\kappa))$ :

$$Im(\lambda(\kappa)) = -\frac{1}{area(\Omega)} \int_{\Omega} v(\kappa, y, z) \overline{\kappa i A \chi(y, z) v(\kappa, y, z)} dydz,$$

where  $v(\kappa, y, z)$  is the unit eigenvector for  $\lambda(\kappa)$ , and we have used Parseval's identity. Continuing, we get

$$Im(\lambda(\kappa)) = i\kappa A \frac{1}{area(\Omega)} \int_{\Omega} v(\kappa, y, z) \chi(y, z) \overline{v(\kappa, y, z)} dydz,$$

so that

$$|Im\lambda(\kappa)| \leq |\kappa|A\|\chi\|_{L^\infty}.$$

Furthermore, since we assumed  $|\kappa| \leq \kappa_0$ , we have that

$$|Im\lambda(\kappa)| \leq |\kappa|A\|\chi\|_{L^\infty} < \frac{\mu_1}{2}.$$

This completes the proof of the Lemma. ■

We now prove Proposition 4.7

**Proof.** First, we prove item (i). Note this separation between the eigenvalues is true for  $\mathcal{B}_0$ , since 0 is an eigenvalue of  $\mathcal{B}_0$  and all other eigenvalues satisfy  $-\mu_n \leq -\mu_1 < 0$ . Let  $\kappa$  satisfy  $|\kappa| \leq \kappa_0$ . To establish this separation for  $\mathcal{B}(\kappa)$ , we will use Kato's definition of a "gap" between operators [Kat95]. Given two operators  $T$  and  $S$  and a closed curve  $\Gamma_* \subset \mathbb{C}$  that separates the spectrum of  $T$  in two parts (one part inside  $\Gamma_*$  and one part outside  $\Gamma_*$ ), if the gap  $\hat{\delta}(T, S)$  is sufficiently small, the closed curve  $\Gamma_*$  also separates the spectrum of  $S$ . The definition of  $\hat{\delta}(T, S)$  is

$$\hat{\delta}(T, S) := \max \left\{ \sup_{u \in G(T), \|u\|=1} dist(u, G(S)), \sup_{v \in G(S), \|v\|=1} dist(v, G(T)) \right\}, \quad (4.14)$$

where  $G(L) = \{(u, Lu) : u \in D(L)\}$  is the graph of the operator  $L$  with domain  $D(L)$ , with  $\|u\|$  being the graph norm. If  $\hat{\delta}(T, S)$  satisfies

$$\hat{\delta}(T, S) < \min_{z \in \Gamma_*} \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + \|(T - z)^{-1}\|^2}}, \quad (4.15)$$

then the closed curve  $\Gamma_*$  also separates the spectrum of  $S$ . See [Kat95, ChapterIV, §3.4, Theorem 3.16]. Before we use Kato's gap, recall from Lemma 4.8 that the eigenvalues  $\lambda(\kappa)$  of  $\mathcal{B}(\kappa)$  and eigenvalues  $\Gamma(\kappa)$  of  $\mathcal{C}(\kappa)$  are in one-to-one correspondence via  $\lambda(\kappa) = \Gamma(\kappa) - \nu^2 \kappa^2$ . We can therefore apply Kato's gap and prove a decomposition for the spectrum of  $\mathcal{C}(\kappa)$ , and get an analogous decomposition using the correspondence between the eigenvalues of  $\mathcal{B}(\kappa)$  and  $\mathcal{C}(\kappa)$ . The advantage of doing so is in the relative simplicity of the operator  $\mathcal{C}(\kappa)$ . Proceeding, we will show that the gap  $\hat{\delta}(\mathcal{C}(\kappa), \mathcal{B}_0)$  satisfies

$$\hat{\delta}(\mathcal{C}(\kappa), \mathcal{B}_0) < \min_{z \in \Gamma_*} \left( \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + \|\mathcal{B}_0 - z\|^{-1}\|^2}} \right)$$

where  $\Gamma_*$  is the boundary of the rectangle  $\{z = x + iy : |x|, |y| \leq \mu_1/2\}$ . If this holds, we will have shown that there is one eigenvalue, which we denote  $\lambda_0(\kappa)$ , that lies inside  $\Gamma_*$  (and hence near 0), and the rest of the eigenvalues lie outside  $\Gamma_*$ .

Let's proceed by computing the gap  $\hat{\delta}(\mathcal{C}(\kappa), \mathcal{B}_0)$ . Using the definition (4.14), we first need to bound

$$\sup_{\|\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{C}(\kappa)\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1} dist\left((\hat{V}, \mathcal{C}(\kappa)\hat{V}), G(\mathcal{B}_0)\right).$$

Pick a  $\hat{V} \in \mathbb{C} \times \ell^2(\mathbb{C})$  with  $\|\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{C}(\kappa)\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1$ . Then

$$\begin{aligned} dist((\hat{V}, \mathcal{C}(\kappa)\hat{V}), G(\mathcal{B}_0)) &\leq \|(\hat{V}, \mathcal{C}(\kappa)\hat{V}) - (\hat{V}, \mathcal{B}_0\hat{V})\| \\ &= \|(0, \mathcal{C}(\kappa)\hat{V} - \mathcal{B}_0\hat{V})\| \\ &= \|\kappa\mathcal{B}_1\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} \\ &\leq |\kappa| (A\|\chi\|_{L^\infty_\Omega}) \|\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} \\ &\leq |\kappa|A\|\chi\|_{L^\infty_\Omega}. \end{aligned}$$

Since this holds for all  $\hat{V}$  with  $\|\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{C}(\kappa)\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1$ , it follows that

$$\sup_{\|\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{C}(\kappa)\hat{V}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1} \text{dist} \left( (\hat{V}, \mathcal{C}(\kappa)\hat{V}), G(\mathcal{B}_0) \right) \leq |\kappa| (A\|\chi\|_{L^\infty}).$$

Next, using the definition of the Kato gap (4.14), we need to bound

$$\sup_{\|\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{B}(0)\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1} \text{dist} \left( (\hat{W}, \mathcal{B}(0)\hat{W}), G(\mathcal{C}(\kappa)) \right).$$

Pick a  $\hat{W} \in \mathbb{C} \times \ell^2(\mathbb{C})$  with  $\|\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{B}_0\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1$ . Then

$$\begin{aligned} \text{dist} \left( (\hat{W}, \mathcal{B}_0\hat{W}), G(\mathcal{C}(\kappa)) \right) &\leq \|(\hat{W}, \mathcal{B}_0\hat{W}) - (\hat{W}, \mathcal{C}(\kappa)\hat{W})\| \\ &= \|(0, \mathcal{B}_0\hat{W} - \mathcal{C}(\kappa)\hat{W})\| \\ &= \|\kappa\mathcal{B}_1\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} \\ &\leq |\kappa| (A\|\chi\|_{L^\infty}) \|\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} \\ &\leq |\kappa| A\|\chi\|_{L^\infty}. \end{aligned}$$

Since this holds for all  $\hat{W} \in \mathbb{C} \times \ell^2(\mathbb{C})$  with  $\|\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{B}_0\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1$ , it follows that

$$\sup_{\|\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} + \|\mathcal{B}(0)\hat{W}\|_{\mathbb{C} \times \ell^2(\mathbb{C})} = 1} \text{dist} \left( (\hat{W}, \mathcal{B}(0)\hat{W}), G(\mathcal{C}(\kappa)) \right) \leq |\kappa| A\|\chi\|_{L^\infty}.$$

Therefore

$$\hat{\delta}(\mathcal{C}(\kappa), \mathcal{B}_0) \leq |\kappa| A\|\chi\|_{L^\infty}. \quad (4.16)$$

Next, we bound

$$\min_{z \in \Gamma_*} \frac{1}{2} \left( \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + \|(\mathcal{B}_0 - z)^{-1}\|^2}} \right)$$

from below. First notice that, for  $z \in \Gamma_*$ ,

$$\|(\mathcal{B}_0 - z)^{-1}\| \leq \frac{2}{\mu_1}.$$

This is true since  $\mathcal{B}_0$  is self adjoint, and  $\mu_1/2$  is the distance from  $\Gamma_*$  to  $\sigma(\mathcal{B}_0)$  [Kat95, Chapter V, §3.5, (3.16)]. Therefore, for all  $z \in \Gamma_*$ ,

$$\frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}} \leq \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + \|(\mathcal{B}_0 - z)^{-1}\|^2}}.$$

Hence

$$\min_{z \in \Gamma_*} \left( \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}} \right) \leq \min_{z \in \Gamma_*} \left( \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + \|(\mathcal{B}_0 - z)^{-1}\|^2}} \right).$$

Next, we compute

$$\min_{z \in \Gamma_*} \left( \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}} \right) = \frac{1}{2} \left( \frac{1}{1 + \frac{\mu_1^2}{2}} \right) \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}}.$$

Therefore, using (4.16), we only need  $\kappa$  to satisfy

$$|\kappa|A\|\chi\|_{L^\infty_\Omega} < \frac{1}{2} \left( \frac{1}{1 + \frac{\mu_1^2}{2}} \right) \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}}.$$

This follows since  $|\kappa| \leq \kappa_0$ . Therefore

$$\hat{\delta}(\mathcal{C}(\kappa), \mathcal{B}_0) < \min_{z \in \Gamma_*} \left( \frac{1}{2} \frac{1}{1 + |z|^2} \frac{1}{\sqrt{1 + \|(\mathcal{B}_0 - z)^{-1}\|^2}} \right)$$

as desired.

This proves that the rectangle  $\Gamma_*$  separates the eigenvalues of  $\mathcal{C}(\kappa)$ . Now suppose that  $\Gamma(\kappa)$  is an eigenvalue not contained in  $\Gamma_*$  (and hence, by the relationship between the spectra of  $\mathcal{B}(\kappa)$  and  $\mathcal{C}(\kappa)$  it corresponds to an eigenvalue  $\lambda(\kappa) \in \Sigma(\kappa)$ .) Then either

- (a)  $\text{Re}(\Gamma(\kappa)) \leq -\mu_1/2$ , or
- (b)  $-\mu_1/2 \leq \text{Re}(\Gamma(\kappa)) \leq 0$ , and  $|\text{Im}(\Gamma(\kappa))| > \mu_1/2$ ,

because Lemma 4.11 implies that none of the eigenvalues of  $\mathcal{C}(\kappa)$  can have positive real part. If case (b) held, then there would be a corresponding eigenvalue  $\lambda(\kappa)$  of  $\mathcal{B}(\kappa)$  with  $|\text{Im}(\lambda(\kappa))| > \mu_1/2$ , and this would violate Lemma 4.11 (ii). Hence case (a) applies and this in turn implies the bound in Proposition 4.7 (i).

Next, we prove item (ii) in Proposition 4.7. Note that, because  $\lambda_0(\kappa)$  is a perturbation of the simple eigenvalue 0 of  $\mathcal{B}_0$ , both  $\lambda_0(\kappa)$  and its spectral projection  $P_0(\kappa)$  perturb smoothly in  $\kappa$  [Kat95]. However, due to Lemma 4.8, we can instead estimate the leading ( $\nu$ -independent) eigenvalue  $\Gamma_0(\kappa)$  of  $\mathcal{C}(\kappa)$ , which is still a perturbation of the simple eigenvalue 0 of  $\mathcal{B}_0$ . We expand this eigenvalue

$$\Gamma_0(\kappa) = \Gamma_0 + \Gamma_1\kappa + \Gamma_2\kappa^2 + \mathcal{O}(\kappa^3) \tag{4.17}$$

and its corresponding eigenvector

$$\hat{V}(\kappa) = \hat{V}_0 + \hat{V}_1\kappa + \hat{V}_2\kappa^2 + \mathcal{O}(\kappa^3), \tag{4.18}$$

where

$$\hat{V}(\kappa) = \begin{pmatrix} \hat{u}_0(k) \\ \hat{U}(\kappa) \end{pmatrix}, \quad \hat{V}_j = \begin{pmatrix} \hat{u}_0^j \\ \hat{U}^j \end{pmatrix}.$$

Now the eigenvalue problem reads

$$\mathcal{C}(\kappa)\hat{V}(\kappa) = \Gamma(\kappa)\hat{V}(\kappa), \tag{4.19}$$

Plugging (4.17) and (4.18) into (4.19), we find

$$\mathcal{B}_0\hat{V}_0 = 0 \cdot \hat{V}_0, \quad \Rightarrow \quad \Gamma_0 = 0, \quad \hat{V}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Next, we find

$$\begin{aligned} \mathcal{B}_1\hat{V}_0 + \mathcal{B}_0\hat{V}_1 &= \Gamma_1\hat{V}_0 \\ \mathcal{B}_1\hat{V}_1 + \mathcal{B}_0\hat{V}_2 &= \Gamma_2\hat{V}_0 + \Gamma_1\hat{V}_1 \\ \mathcal{B}_1\hat{V}_2 + \mathcal{B}_0\hat{V}_3 &= \Gamma_3\hat{V}_0 + \Gamma_2\hat{V}_1 + \Gamma_1\hat{V}_2 \end{aligned} \tag{4.20}$$

and so on. Solving the first equation, we find

$$\Gamma_1 = 0, \quad \hat{V}_1 = \begin{pmatrix} c_1 \\ iA\Upsilon^{-1}\tilde{\chi} \end{pmatrix},$$

where the scalar constant  $c_1$  is undetermined but can be fixed by normalizing the eigenvectors. At  $\mathcal{O}(\kappa^2)$ , we similarly find

$$\Gamma_2 = -D_{td}, \quad \hat{V}_2 = \begin{pmatrix} c_2 \\ iAc_1\Upsilon^{-1}\tilde{\chi} - A^2\Upsilon^{-1}[\tilde{\chi} * (\Upsilon^{-1}\tilde{\chi})] \end{pmatrix}.$$

Finally, at  $\mathcal{O}(\kappa^3)$ , the first component in the equation implies

$$\begin{aligned} \Gamma_3 &= c_1(D_{td}) + iA\tilde{\chi} \cdot [iAc_1\Upsilon^{-1}\tilde{\chi} - A^2[\tilde{\chi} * (\Upsilon^{-1}\tilde{\chi})]] \\ &= -iA^3\tilde{\chi} \cdot [\tilde{\chi} * (\Upsilon^{-1}\tilde{\chi})]. \end{aligned}$$

In particular,  $\Gamma_3$  is purely imaginary, and therefore

$$\Gamma_0(k) = -D_{td}\kappa^2 + ir\kappa^3 + \mathcal{O}(\kappa^4),$$

where

$$r = -A^3\tilde{\chi} \cdot [\tilde{\chi} * (\Upsilon^{-1}\tilde{\chi})]. \quad (4.21)$$

Finally, using Lemma 4.8, we have

$$\begin{aligned} \lambda_0(\kappa) &= -(\nu^2 + D_{td})\kappa^2 + \Lambda_0(\kappa) \\ &= -\nu_{td}\kappa^2 + \Lambda_0(\kappa), \end{aligned}$$

where  $\Lambda_0(\kappa) = ir\kappa^3 + \mathcal{O}(\kappa^4)$  is independent of  $\nu$ . This completes the proof of item (ii), and of Proposition 4.7.  $\blacksquare$

#### 4.1.2 High wavenumber estimates using standard diffusive estimates

Next, we consider the behavior of the spectrum of  $\mathcal{B}(\kappa)$  for large  $|\kappa|$ .

**Corollary 4.12.** *Given any fixed constant  $\kappa_1$ , for all  $|\kappa| \geq \frac{\kappa_1}{\nu}$  we have*

$$\|e^{\mathcal{B}(\kappa)T}W\|_Y \leq Ce^{-\kappa_1^2 T} \|W\|_Y.$$

**Proof.** This follows immediately from Lemma 4.11, using the fact that  $\mathcal{B}(\kappa)$  generates an analytic semi-group.  $\blacksquare$

#### 4.1.3 Intermediate wavenumber estimates via hypocoercivity

In this subsection, we prove the following Lemma.

**Proposition 4.13.** *Let  $\kappa_0 = \min \left\{ \frac{\mu_1}{2}, \frac{1}{2A\|\tilde{\chi}\|_{L^\infty_\Omega}} \left( \frac{1}{1+\frac{\mu_1}{2}} \right) \frac{1}{\sqrt{1+(\frac{2}{\mu_1})^2}} \right\}$ . Then there exist positive constants  $\kappa_1$  and  $\delta \in (0, \frac{1}{4})$  such that for all  $\frac{\kappa_1}{\nu} \geq |\kappa| \geq \kappa_0(1-\delta)$  and  $T > 0$ , we have*

$$\|e^{\mathcal{B}(\kappa)T}W\|_Y \leq Ce^{-MT} \|W\|_Y,$$

where  $M$  and  $C$  are constants that are independent of  $\nu$ .

**Remark 4.14.** *This result does not appear to be obvious. A naive estimate, such as that in the proof of Corollary 4.12, would only give*

$$\|e^{\mathcal{B}(\kappa)T}W\|_Y \leq Ce^{-\nu^2(\kappa_0(1-\delta))^2T}\|W\|_Y.$$

For large times  $T = 1$ , this does not actually produce decay:  $e^{-\nu^2(\kappa_0(1-\delta))^2T} = e^{-\nu^2(\kappa_0(1-\delta))^2} \sim 1$ . Therefore, we really do need the stronger result given in Proposition 4.13 to conclude that small wavenumbers  $|\kappa| \leq \kappa_0$  really do give the leading order behavior of solutions.

**Proof.** Let  $\delta \in (0, \frac{1}{4})$  and fix  $\kappa \in [\kappa_0(1-\delta), \kappa_1/\nu]$ , with any fixed  $\kappa_1 > \nu\kappa_0(1-\delta)$ . We will study the decay of solutions to

$$\frac{d}{dT}\hat{U} = \mathcal{B}(\kappa)\hat{U},$$

with  $\hat{U}$  and  $\mathcal{B}(\kappa)$  defined in (4.3) and (4.5) using Villani's theory of hypocoercivity [Vil09]. Writing this equation in components and writing  $\hat{u}_0^s = u$  and  $\hat{u}_m^s = v_m$  with  $m = 1, 2, \dots$  for notational convenience, we have

$$\begin{aligned} \partial_T u &= -\nu^2\kappa^2 u + Ai\kappa \sum_{m=1}^{\infty} \chi_m v_m \\ \partial_T v_m &= -(\nu^2\kappa^2 + \mu_m)v_m + Ai\kappa\chi_m u + Ai\kappa \sum_{j=1}^{\infty} \chi_{m,j} v_j. \end{aligned}$$

Motivated by [Vil09], we consider the functional

$$\Phi[(u, v)](T) = \zeta_0 u \bar{u} + \sum_m \zeta_m v_m \bar{v}_m + 2\text{Re} \left( iu \sum_m \sigma_m \bar{v}_m \right)$$

with  $\zeta_0$ ,  $\zeta_m$ , and  $\sigma_m$  to be defined below. We will show that  $\dot{\Phi} \leq -M\Phi$  for some constant  $M$  that is independent of  $\nu$  and  $\kappa$ , as long as  $\kappa_1/\nu \geq |\kappa| \geq \kappa_0(1-\delta)$ . We will also chose  $\zeta_0$ ,  $\zeta_m$ , and  $\sigma_m$  so that there exist constants  $c_{1,2}$  independent of  $\nu$  and  $\kappa$  so that  $c_1\|(u, v)\|_Y \leq \Phi(u, v) \leq c_2\|(u, v)\|_Y$ . This will imply that  $\|(u, v)(\tau)\|_Y \leq (c_2/c_1)e^{-MT}$ . Undoing the scalings will then imply the decay claimed in the Proposition.

We compute

$$\begin{aligned} \dot{\Phi} &= -2\zeta_0\nu^2\kappa^2|u|^2 - 2A\kappa\zeta_0\text{Re} \left( iu \sum_m \chi_m \bar{v}_m \right) - 2 \sum_m \zeta_m(\nu^2\kappa^2 + \mu_m)|v_m|^2 + 2A\kappa\text{Re} \left( iu \sum_m \zeta_m \chi_m \bar{v}_m \right) \\ &\quad + 2A\kappa\text{Re} \left( i \sum_m \zeta_m \bar{v}_m \sum_{j=1}^{\infty} \chi_{m,j} v_j \right) - 2\nu^2\kappa^2\text{Re} \left( iu \sum_m \sigma_m \bar{v}_m \right) - 2A\kappa\text{Re} \left( \sum_j \chi_j v_j \sum_m \sigma_m \bar{v}_m \right) \\ &\quad - 2\text{Re} \left( iu \sum_m \sigma_m(\nu^2\kappa^2 + \mu_m)\bar{v}_m \right) + 2A\kappa\text{Re} \left( |u|^2 \sum_m \sigma_m \chi_m \right) \\ &\quad + 2A\kappa\text{Re} \left( u \sum_m \sigma_m \sum_{j=1}^{\infty} \chi_{m,j} \bar{v}_j \right). \end{aligned}$$

Next, define

$$\sigma_m = -\frac{c}{2A\kappa\mu_m}\chi_m, \quad \zeta_m = \zeta_0 \quad \forall m,$$

where  $c$  is a constant to be determined. Note that this choice of  $\zeta_m$  implies

$$-2A\kappa\zeta_0\operatorname{Re}\left(iu\sum_m\chi_m\bar{v}_m\right)+2A\kappa\operatorname{Re}\left(iu\sum_m\zeta_m\chi_m\bar{v}_m\right)=0$$

Also,

$$2A\kappa\operatorname{Re}\left(i\sum_m\zeta_m\bar{v}_m\sum_{j=1}^{\infty}\chi_{m,j}v_j\right)=0,$$

which results from the fact that the  $\chi_{m,j}$  are real and  $\chi_{m,j}=\chi_{j,m}$ . This follows from the fact that the eigenfunctions  $\psi_j$  of the Laplacian on the cross section  $\Omega$  can be chosen to be real. Therefore, we have

$$\begin{aligned}\dot{\Phi} &= -2\zeta_0\nu^2\kappa^2|u|^2-2\zeta_0\sum_m(\nu^2\kappa^2+\mu_m)|v_m|^2+\frac{c\nu^2\kappa}{A}\operatorname{Re}\left(iu\sum_m\frac{1}{\mu_m}\chi_m\bar{v}_m\right)+c\operatorname{Re}\left(\sum_j\chi_jv_j\sum_m\frac{1}{\mu_m}\chi_m\bar{v}_m\right) \\ &\quad +\frac{c}{A}\operatorname{Re}\left(iu\sum_m\frac{\chi_m}{\kappa\mu_m}(\nu^2\kappa^2+\mu_m)\bar{v}_m\right)-c|u|^2|\chi|_{\mu}^2-c\operatorname{Re}\left(u\sum_m\frac{\chi_m}{\mu_m}\sum_{j=1}^{\infty}\chi_{m,j}\bar{v}_j\right) \\ &\leq\left[-2\zeta_0\nu^2\kappa^2+\frac{c\nu^2|\kappa|}{AQ_1^2}+\frac{c}{2A|\kappa|Q_2^2}-c|\chi|_{\mu}^2+\frac{c}{2Q_3^2}\right]|u|^2 \\ &\quad \left[-2\zeta_0(\mu_1+\nu^2\kappa^2)+\frac{c\nu^2|\kappa|Q_1^2|\chi|_{\mu}^2}{A}+c|\chi||\chi|_{\mu}+ \frac{cQ_2^2}{2A|\kappa|}|\chi|^2+\frac{cQ_3^2}{2}|\chi|_{\mu}^2|\chi|_{L^{\infty}}^2\right]|v|^2. \\ &= (I_u+II_u)|u|^2+(I_v+II_v)|v|^2,\end{aligned}$$

where we denote  $|v|=\|v\|_{\ell^2}$  and where

$$I_u=-c|\chi|_{\mu}^2+\frac{c}{2A|\kappa|Q_2^2}+\frac{c}{2Q_3^2},\quad II_u=-2\zeta_0\nu^2\kappa^2+\frac{c\nu^2|\kappa|}{AQ_1^2},$$

and

$$I_v=-2\zeta_0\mu_1+c|\chi||\chi|_{\mu}+\frac{cQ_3^2}{2}|\chi|_{\mu}^2|\chi|_{L^{\infty}}^2+\frac{cQ_2^2}{2A|\kappa|}|\chi|^2,\quad II_v=-2\zeta_0\nu^2\kappa^2+\frac{c\nu^2|\kappa|Q_1^2|\chi|_{\mu}^2}{A}.$$

Recall that  $0<\delta<1/4$ ,  $|\kappa|>\kappa_0(1-\delta)$ , where  $\kappa_0=\min\left\{\frac{\mu_1}{2},\frac{1}{2A\|\chi\|_{L^{\infty}}}\left(\frac{1}{1+\frac{\mu_1^2}{2}}\right)\frac{1}{\sqrt{1+(\frac{2}{\mu_1})^2}}\right\}$ . Furthermore, let

$$c<\min\left\{\frac{1-\delta}{|\chi|_{\mu}^2},\frac{\mu_1}{|\chi||\chi|_{\mu}+|\chi|_{L^{\infty}}^2+\frac{2|\chi|^2}{3A^2\kappa_0^2|\chi|_{\mu}^2}},A^2\kappa_0^2(1-\delta),\frac{12\mu_1}{|\chi|_{\mu}^2}\right\}.$$

We choose  $\zeta_0=1$ ,  $Q_1^2=Q_2^2=\frac{1}{A\kappa_0|\chi|_{\mu}^2}$  and  $Q_3^2=\frac{2}{|\chi|_{\mu}^2}$ . Then

$$I_u=c|\chi|_{\mu}^2\left(-\frac{3}{4}+\frac{\kappa_0}{2|\kappa|}\right)\leq-\frac{c|\chi|_{\mu}^2}{12}\leq-\mu_1$$

since  $|\kappa|>\kappa_0(1-\delta)$ ,  $0<\delta<1/4$ , and  $c<12\mu_1/|\chi|_{\mu}^2$ . Next, notice that the above choices imply that

$$II_u=\nu^2(-2\kappa^2+c|\chi|_{\mu}^2\kappa_0|\kappa|)\leq-\nu^2\kappa^2,$$

where we have used the facts that  $|\kappa| > \kappa_0(1 - \delta)$  and  $c < \frac{(1-\delta)}{|\chi|_\mu^2}$ . Similarly,

$$I_v = -2\mu_1 + c \left[ |\chi| |\chi|_\mu + |\chi|_{L^\infty}^2 + \frac{2|\chi|^2}{3A^2\kappa_0^2|\chi|_\mu^2} \right] \leq -\mu_1.$$

Finally,

$$II_v = \nu^2 \left( -2\kappa^2 + \frac{c}{A^2\kappa_0} |\kappa| \right) \leq -\nu^2\kappa^2$$

because  $c < a^2\kappa_0^2(1 - \delta)$ . Therefore

$$\dot{\Phi} \leq -(\mu_1 + \nu^2\kappa^2)(|u|^2 + |v|^2).$$

Also, we have that

$$\begin{aligned} \Phi &\leq \left( 1 + \frac{c}{2A|\kappa|} \right) |u|^2 + \left( 1 + \frac{c|\chi|_\mu^2}{2A|\kappa|} \right) |v|^2 \\ &\leq \tilde{M}(|u|^2 + |v|^2), \end{aligned}$$

where  $\tilde{M} = 1 + \frac{A\kappa_0}{2} \max\{1, |\chi|_\mu^2\}$ . As a result,

$$\dot{\Phi} \leq -M\Phi, \tag{4.22}$$

where  $M = \mu_1/\tilde{M}$ . If we now additionally require that

$$c \leq \min \left\{ A\kappa_0(1 - \delta), \frac{A\kappa_0(1 - \delta)}{|\chi|_\mu^2} \right\},$$

we find

$$\Phi \geq \left( 1 - \frac{c}{2A\kappa_0(1 - \delta)} \right) |u|^2 + \left( 1 - \frac{c|\chi|_\mu^2}{2A\kappa_0(1 - \delta)} \right) |v|^2 \geq \frac{1}{2}(|u|^2 + |v|^2).$$

Therefore,

$$|u(T)|^2 + |v(T)|^2 \leq 2\Phi(T) \leq 2e^{-MT}\Phi(0) \leq 4e^{-MT}[|u(0)|^2 + |v(0)|^2],$$

which completes the proof of the Proposition. ■

## 4.2 Splitting of the semigroup

The goal of this subsection is to establish the decay rates on the semigroup by splitting it as

$$e^{B(\kappa)(T-s)} = \mathcal{E}_{\text{high}}(\kappa, T-s) + \mathcal{E}_{\text{low}}(\kappa, T-s) + \mathcal{T}_N(\kappa, T-s) + \mathcal{R}_N(\kappa, T-s), \tag{4.23}$$

where the components are defined as follows. Both  $\mathcal{E}_{\text{high,low}}$  will be exponentially decaying pieces that correspond to high and low wavenumbers, respectively. The terms  $\mathcal{T}_N$  and  $\mathcal{R}_N$  will both correspond to the leading order eigenvalue  $\lambda_0(\kappa) = -\nu_{td}\kappa^2 + \Lambda_0(\kappa)$  of  $\mathcal{B}(\kappa)$ , defined in Proposition 4.7, with  $\mathcal{T}_N$  arising from the Taylor diffusion term  $-\nu_{td}\kappa^2$  and  $\mathcal{R}_N$  arising from the remainder  $\Lambda_0(\kappa)$ .



To precisely define each term in (4.23), first let  $\psi(\kappa)$  be a smooth bump function that equals 1 for  $|\kappa| \leq \kappa_0$  and 0 for  $|\kappa| \geq 2\kappa_0$ , where  $\kappa_0$  is any fixed positive number such that, for example,

$$\kappa_0 \leq \frac{1}{2} \min \left\{ \frac{\mu_1}{2}, \frac{1}{2A\|\chi\|_{L^\infty_\Omega}} \left( \frac{1}{1 + \frac{\mu_1^2}{2}} \right) \frac{1}{\sqrt{1 + (\frac{2}{\mu_1})^2}} \right\}$$

as motivated by Proposition 4.7. (We will need to adjust the value of  $\kappa_0$ , below, to make it slightly smaller.) Furthermore, let  $P_0(\kappa)$  be the ( $\nu$ -independent) projection onto the eigenspace for the leading eigenvalue  $\lambda_0(\kappa)$  of  $\mathcal{B}(\kappa)$ , defined in Proposition 4.7, and let  $Q_0(\kappa) = I - P_0(\kappa)$  be its complement. We can then define

$$\mathcal{E}_{\text{high}}(\kappa, T-s) = (1 - \psi(\kappa))e^{\mathcal{B}(\kappa)(T-s)} \quad (4.24)$$

$$\mathcal{E}_{\text{low}}(\kappa, T-s) = \psi(\kappa)Q_0(\kappa)e^{\mathcal{B}(\kappa)(T-s)}. \quad (4.25)$$

We use a Taylor expansion to define the remaining two terms  $\mathcal{T}_N$ , acting on a function  $\hat{G}(\kappa, s)$ , and  $\mathcal{R}_N$  as

$$\mathcal{T}_N(\kappa, T-s)\hat{G}(\kappa, s) = e^{-\nu_{td}\kappa^2(T-s)} \sum_{\ell=0}^N \frac{1}{\ell!} \partial_\kappa^\ell \left( \psi(\kappa)P_0(\kappa)e^{\Lambda_0(\kappa)(T-s)}\hat{G}(\kappa, s) \right) \Big|_{\kappa=0} \kappa^\ell \quad (4.26)$$

$$\mathcal{R}_N(\kappa, T-s) = e^{-\nu_{td}\kappa^2(T-s)}\psi(\kappa)P_0(\kappa)e^{\Lambda_0(\kappa)(T-s)} - \mathcal{T}_N(\kappa, T-s). \quad (4.27)$$

With this definition, we have

$$\mathcal{T}_N(\kappa, T-s) + \mathcal{R}_N(\kappa, T-s) = \psi(\kappa)P_0(\kappa)e^{\mathcal{B}(\kappa)(T-s)} = \psi(\kappa)P_0(\kappa)e^{\lambda_0(\kappa)(T-s)} = e^{-\nu_{td}\kappa^2(T-s)}\psi(\kappa)P_0(\kappa)e^{\Lambda_0(\kappa)(T-s)}.$$

We now obtain decay estimates on each piece of (4.23).

#### 4.2.1 Bounds on $\mathcal{E}_{\text{low}}$

Before providing bounds on  $\mathcal{E}_{\text{low}}$ , we first state the following lemma.

**Lemma 4.15.** *Recall  $\nu_{td} = \nu^2 + D_{td}$ , where  $D_{td} = A^2\|\chi\|_\mu^2$ . Let  $d > 0$  and  $T > 0$ . Then*

$$\|\kappa^d e^{-\nu_{td}\kappa^2(1+T)}\|_{L^2(\mathbb{R})} \leq C(1+T)^{-\frac{d}{2}-\frac{1}{4}},$$

where the constant  $C = C(d)$  is independent of  $\nu$ .

**Proof.** This follows from a direct calculation; the  $\nu$ -independence of the constant  $C$  follows from the fact that

$$\nu_{td}^{-1} = (\nu^2 + D_{td})^{-1} \leq D_{td}^{-1}$$

■

We now prove the following lemma, which provides estimates on  $\mathcal{E}_{\text{low}}$ . Recall that the norm  $\|\cdot\|$  is defined in (4.9).

**Lemma 4.16.** (i)  $\|\mathcal{E}_{\text{low}}(\cdot, T)\hat{V}(\cdot)\| \leq Ce^{-\frac{\mu_1}{2}T}\|\hat{V}(\cdot)\|$

$$(ii) \quad \left\| \int_0^T \mathcal{E}_{\text{low}}(\cdot, T-s) \hat{F}(\cdot, s) ds \right\| \leq C(1+T)^{-\frac{N}{6}-\frac{1}{12}}.$$

**Proof.** By (4.9) and Corollary 4.10, we have

$$\begin{aligned} \|\mathcal{E}_{\text{low}}(\cdot, T) \hat{V}(\cdot)\|^2 &= \int_{\mathbb{R}} \|\mathcal{E}_{\text{low}}(\kappa, T) \hat{V}(\kappa)\|_Y^2 d\kappa \\ &\leq \int_{\mathbb{R}} C e^{-\mu_1 T} \|\hat{V}(\kappa)\|_Y^2 d\kappa = C e^{-\mu_1 T} \|\hat{V}(\cdot)\|^2. \end{aligned}$$

This proves (i). To prove item (ii), note

$$\begin{aligned} \left\| \int_0^T \mathcal{E}_{\text{low}}(\cdot, T-s) \hat{F}(\cdot, s) ds \right\| &= \left\| \int_0^T \mathcal{E}_{\text{low}}(\cdot, T-s) \hat{F}(\cdot, s) ds \right\|_Y \|L^2(\mathbb{R})\| \\ &\leq \int_0^T C e^{-\frac{\mu_1}{2}(T-s)} \|\hat{F}(\cdot, s)\|_Y \|L^2(\mathbb{R})\| ds. \end{aligned}$$

Now from Lemma 4.4, we know that

$$\|\hat{F}(\kappa, s)\|_Y \leq C(1+s)^{\frac{N-1}{2}-\frac{1}{2}(j+n)} e^{-\nu t \kappa^2(1+s)} [|\kappa|^{N+1} + |\kappa|^{N+2}(1+s)^{1/2} + |\kappa|^{N+2} + |\kappa|^{N+3}(1+s)^{1/2}].$$

Next, using Lemma 4.15, we have that

$$\begin{aligned} \|\|\hat{F}(\cdot, s)\|_Y\|_{L^2(\mathbb{R})} &\leq C(1+s)^{\frac{N-1}{2}-\frac{N+1}{2}-\frac{1}{4}}(1+s)^{-\frac{1}{2}(j+n)} \\ &= C(1+s)^{-\frac{5}{4}}(1+s)^{-\frac{1}{2}(j+n)}, \end{aligned}$$

where the constant  $C$  is independent of  $\nu$ . Therefore

$$\begin{aligned} \left\| \int_0^T \mathcal{E}_{\text{low}}(\cdot, T-s) \hat{F}(\cdot, s) ds \right\| &\leq \int_0^T C e^{-\frac{\mu_1}{2}(T-s)} (1+s)^{-\frac{5}{4}} (1+s)^{-\frac{1}{2}(j+n)} ds \\ &\leq C(1+T)^{-\frac{1}{2}(j+n)-\frac{1}{4}} \\ &\leq C(1+T)^{-\frac{N}{6}-\frac{1}{12}}, \end{aligned}$$

where the exponent in the last line follows from the fact that  $N-1=3j+n$ , and  $n \in \{0, 1, 2\}$ . ■

## 4.2.2 Bounds on $\mathcal{E}_{\text{high}}$

We prove the following Lemmas.

**Lemma 4.17.** *There exist constants  $C$  and  $M_1$ , independent of  $\nu$ , such that*

$$(i) \quad \|\mathcal{E}_{\text{high}}(\cdot, t) \hat{V}(\cdot)\| \leq C e^{-M_1 T} \|\hat{V}(\cdot)\|$$

(ii)

$$\left\| \int_0^T \mathcal{E}_{\text{high}}(\cdot, T-s) \hat{F}(\cdot, s) ds \right\| \leq C e^{-\frac{1}{4} M_1 T}.$$

**Proof.** We can use Proposition 4.13 for  $\kappa_0 \leq |\kappa| \leq \frac{\kappa_1}{\nu}$  and Corollary 4.12 for  $|\kappa| \geq \frac{\kappa_1}{\nu}$  to find

$$\|e^{\mathcal{B}(\kappa)T} W\|_Y \leq C e^{-MT} \|W\|_Y$$

for all  $|\kappa| \geq \kappa_0$ . Therefore, we have

$$\begin{aligned} \|\mathcal{E}_{\text{high}}(\cdot, T)\hat{V}(\cdot)\|^2 &= \int_{\mathbb{R}} \|\mathcal{E}_{\text{high}}(\kappa, T)\hat{V}(\kappa)\|_Y^2 d\kappa \\ &\leq \int_{\mathbb{R}} C e^{-2MT} \|\hat{V}(\kappa)\|_Y^2 d\kappa = C e^{-2MT} \|\hat{V}(\cdot)\|^2, \end{aligned}$$

which proves (i). Item (ii) follows additionally from Lemma 4.4 and the estimate

$$\begin{aligned} \left\| \int_0^T \mathcal{E}_{\text{high}}(\cdot, T-s)\hat{F}(\cdot, s) ds \right\| &\leq \int_0^T C e^{-M(T-s)} \|(1-\psi(\cdot))\|\hat{F}(\cdot, s)\|_{L^2(\mathbb{R})} ds \\ &\leq \int_0^T C e^{-M(T-s)} (1+s)^{\frac{N-1-n}{3}} \|(1-\psi(\kappa))|\kappa|^{(N+1)} e^{-\nu_{td}\kappa^2(1+s)} [1+(|\kappa|+|\kappa|^2)(1+s)^{1/2}]\|_{L^2(\mathbb{R})} ds \\ &\leq \int_0^T C e^{-M(T-s)} \sup_{|\kappa| \geq 2\kappa_0} (e^{-\frac{\nu_{td}}{2}\kappa^2(1+s)})(1+s)^{\frac{N-1-n}{3}} \| |\kappa|^{(N+1)} e^{-\frac{\nu_{td}}{2}\kappa^2(1+s)} [1+(|\kappa|+|\kappa|^2)(1+s)^{1/2}] \|_{L^2(\mathbb{R})} ds \\ &\leq \int_0^T C e^{-M(T-s)} \sup_{|\kappa| \geq 2\kappa_0} (e^{-\frac{\nu_{td}}{2}\kappa^2(1+s)})(1+s)^{-\frac{N}{6}-\frac{1}{2}} ds \\ &= \int_0^T C e^{-M(T-s)} e^{-2\nu_{td}\kappa_0^2(1+s)} (1+s)^{-\frac{N}{6}-\frac{1}{2}} ds \\ &= \int_0^T C e^{-M(T-s)} e^{-\nu_{td}\kappa_0^2(1+s)} e^{-\nu_{td}\kappa_0^2(1+s)} (1+s)^{-\frac{N}{6}-\frac{1}{2}} ds \\ &\leq C e^{-MT} e^{(M-\nu_{td}\kappa_0^2)T} \leq C e^{-D_{td}\kappa_0^2 T}. \end{aligned}$$

■

### 4.2.3 Bounds on $\mathcal{R}_N$

In this section we prove the following lemma.

**Lemma 4.18.** *Recall that  $\mathcal{R}_N(\kappa, T)$  is defined in (4.27),  $\|\cdot\|$  in (4.9), and  $\hat{F}$  in (4.6). Then*

$$(i) \quad \|\mathcal{R}_N(\cdot, T)\hat{V}(\cdot)\|^2 \leq C(\psi, P_0, \hat{V}) T^{-\frac{N}{6}-\frac{5}{12}}$$

$$(ii) \quad \left\| \int_0^T \mathcal{R}_N(\cdot, T-s)\hat{F}(\cdot, s) ds \right\| \leq C(1+T)^{-\frac{N}{6}-\frac{1}{12}}$$

for all  $T > 0$ , where the constant  $C(\psi, P_0, \hat{V})$  depends on the first  $N+1$  derivatives of  $\psi, P_0$ , and  $\hat{V}$ . In particular, we need to require that  $\|\partial_\kappa^\ell \hat{V}\|$  is bounded for all  $0 \leq \ell \leq N+1$ .

**Remark 4.19.** *We can ensure that the initial condition  $\hat{U}(\kappa, 0)$  in (4.8) has  $\|\partial_\kappa^\ell \hat{U}(\cdot, 0)\|$  bounded for all  $0 \leq \ell \leq N+1$  by requiring that the initial condition  $u(x, y, z, 0)$  to (1.1) lies in  $L^2((N+1))$ .*

**Proof.** To estimate  $\mathcal{R}_N$ , notice that a smooth function minus the first  $N$  terms of its Taylor series can be written

$$f(\kappa) - \sum_{j=0}^N \frac{1}{j!} f^{(j)}(0) \kappa^j = \int_0^\kappa \int_0^{\kappa_N} \cdots \int_0^{\kappa_1} \partial_y^{N+1} f(y) dy d\kappa_1 \cdots d\kappa_N.$$

Therefore, we can write

$$\mathcal{R}_N(\kappa, T-s)\hat{G}(\kappa, T-s) = e^{-\nu_{td}\kappa^2 T} \int_0^\kappa \int_0^{\kappa_N} \cdots \int_0^{\kappa_1} \partial_y^{N+1} \left[ \psi(y) e^{\Lambda_0(y)(T-s)} P_0(y) \hat{G}(y, T-s) \right] dy d\kappa_1 \cdots d\kappa_N.$$

Furthermore, the computation of the expansion of  $\lambda_0(\kappa)$  that follows equation (4.17) implies that

$$|\lambda_0(\kappa) + \nu_{td}\kappa^2| = |\Lambda_0(\kappa)| \leq C|\kappa|^3, \quad \text{for } |\kappa| \leq 2\kappa_0$$

for some constant  $C$  that is independent of  $\nu$ . The  $y$ -derivatives in the above integral expression could fall on any of the terms in the brackets. Thus, we need to bound terms of the form

$$\|(\partial_\kappa^{m_1} \psi(\kappa))(\partial_\kappa^{m_2} e^{\Lambda_0(\kappa)T})(\partial_\kappa^{m_3} P_0(\kappa))(\partial_\kappa^{m_4} \hat{V}(\kappa))\|, \quad m_1 + m_2 + m_3 + m_4 = N + 1.$$

Using the form of  $\Lambda_0(\kappa)$ , we have

$$\partial_\kappa^{m_2} e^{\Lambda_0(\kappa)T} \sim (\kappa^2 T)^{\rho_1} (\kappa T)^{\rho_2} (T)^{\rho_3} (T)^{\rho_4} \dots (T)^{\rho_{m_2}} e^{\Lambda_0(\kappa)T},$$

where  $\rho_1 + 2\rho_2 + \dots + m_2\rho_{m_2} = m_2$  and  $\rho_i \in \{0, 1, \dots, m_2\}$  for all  $i$ . Thus, the  $\rho_i$  term corresponds to  $i$  derivatives falling on  $\Lambda_0(\kappa)$ . Therefore,

$$\begin{aligned} \|\mathcal{R}_N(\cdot, T)\hat{V}(\cdot)\|^2 &\leq \int_{\mathbb{R}} \left\| e^{-\nu_{td}\kappa^2 T} \int_0^\kappa \int_0^{\kappa_N} \dots \int_0^{\kappa_1} \partial_y^{N+1} [\psi(y)e^{\Lambda_0(y)T} P_0(y)\hat{V}(y)] dy d\kappa_1 \dots d\kappa_N \right\|_Y^2 d\kappa \\ &\leq \sum_{m_1+m_2+m_3+m_4=N+1} \sup_{|\kappa| \leq 2\kappa_0} \left\| (\partial_\kappa^{m_1} \psi(\kappa))(\partial_\kappa^{m_3} P_0(\kappa))(\partial_\kappa^{m_4} \hat{V}(\kappa)) \right\|_Y^2 \\ &\quad \times \int_{|\kappa| \leq 2\kappa_0} \left( e^{-\nu_{td}\kappa^2 T} \int_0^\kappa \int_0^{\kappa_N} \dots \int_0^{\kappa_1} \left| \partial_y^{m_2} e^{\Lambda_0(y)T} \right| dy d\kappa_1 \dots d\kappa_N \right)^2 d\kappa \\ &= C(\psi, P_0, \hat{V}) \int_{|\kappa| \leq 2\kappa_0} e^{-2\nu_{td}\kappa^2 T} \\ &\quad \times \left( \int_0^\kappa \int_0^{\kappa_N} \dots \int_0^{\kappa_1} \left| (y^2 T)^{\rho_1} (yT)^{\rho_2} (T)^{\rho_3} (T)^{\rho_4} \dots (T)^{\rho_{m_2}} e^{\Lambda_0(y)T} \right| dy d\kappa_1 \dots d\kappa_N \right)^2 d\kappa \\ &\leq C(\psi, P_0, \hat{V}) T^{2(\rho_1 + \dots + \rho_{m_2})} \int_{|\kappa| \leq 2\kappa_0} e^{-2\nu_{td}\kappa^2 T} e^{2C|\kappa|^3 T} |\kappa|^{2(2\rho_1 + \rho_2 + N + 1)} d\kappa. \end{aligned}$$

The constant  $C(\psi, P_0, \hat{V})$  is determined by  $\sup_{|\kappa| \leq 2\kappa_0} \left\| (\partial_\kappa^{m_1} \psi(\kappa))(\partial_\kappa^{m_3} P_0(\kappa))(\partial_\kappa^{m_4} \hat{V}(\kappa)) \right\|_Y^2$ . The function  $\psi$  and the projection  $P_0$  are smooth, bounded, and independent of  $\nu$ , so we need not worry about derivatives that fall on them. Notice that, for  $z = \kappa\sqrt{T}$ , we have

$$\begin{aligned} \int_{|\kappa| \leq 2\kappa_0} e^{-2\nu_{td}\kappa^2 T} e^{2C|\kappa|^3 T} |\kappa|^\rho d\kappa &= CT^{-\frac{(\rho+1)}{2}} \int_{|z| \leq 2\kappa_0\sqrt{T}} |z|^\rho e^{-2\nu_{td}z^2} e^{2CT|z|/\sqrt{T}|^3} dz \\ &\leq CT^{-\frac{(\rho+1)}{2}} \int_{|z| \leq 2\kappa_0\sqrt{T}} |z|^\rho e^{-\nu_{td}z^2} e^{-z^2\left(\nu_{td} - \frac{2C|z|}{\sqrt{T}}\right)} dz \\ &\leq CT^{-\frac{(\rho+1)}{2}}, \end{aligned}$$

Note that we have used the fact that  $|z| \leq 2\kappa_0\sqrt{T}$  with  $\kappa_0 = \mathcal{O}(1)$ . Therefore, after possibly making  $\kappa_0$  smaller if necessary,  $\nu_{td} - \frac{2C|z|}{\sqrt{T}} \geq 0$ . As a result

$$\|\mathcal{R}_N(\cdot, T)\hat{V}(\cdot)\| \leq C(\psi, P_0, \hat{V}) T^{\rho_1 + \dots + \rho_{m_2}} T^{-\frac{1}{4} - \frac{1}{2}(2\rho_1 + \rho_2 + N + 1)}.$$

Notice that

$$\begin{aligned} \rho_1 + \dots + \rho_{m_2} - \frac{1}{4} - \frac{1}{2}(2\rho_1 + \rho_2 + N + 1) &= \frac{1}{2}\rho_2 + \rho_3 + \rho_4 + \dots + \rho_{m_2} - \frac{3}{4} - \frac{N}{2} \\ &\leq \frac{1}{3}(\rho_1 + 2\rho_2 + \dots + m_2\rho_{m_2}) - \frac{3}{4} - \frac{N}{2} \\ &\leq -\frac{N}{6} - \frac{5}{12}. \end{aligned}$$

In addition,

$$2\rho_1 + 2\rho_2 + 2\rho_3 + 3\rho_4 + \cdots + (m_2 - 1)\rho_{m_2} \leq 2m_1 \leq 2(N + 1).$$

Therefore, we obtain

$$\|\mathcal{R}_N(\cdot, T)\hat{V}(\cdot)\| \leq C(\psi, P_0, \hat{V})T^{-\frac{N}{6}-\frac{5}{12}},$$

which proves (i). To prove (ii), Lemma 4.4 implies

$$\|\hat{F}(\kappa, s)\|_Y \leq C|\kappa|^{N+1}(1+s)^{\frac{N-1}{3}}e^{-\nu_{td}\kappa^2(1+s)}[1 + (|\kappa| + |\kappa|^2)(1+s)^{1/2}].$$

Similarly,

$$\|\partial_k^{m_4}\hat{F}(k, s)\|_Y \leq C|\kappa|^{N+1-r_1}(1+s)^{\frac{N-1}{3}}(\partial_\kappa^{r_2}e^{-\nu_{td}\kappa^2(1+s)})\partial_\kappa^{r_3}[1 + (|\kappa| + |\kappa|^2)(1+s)^{1/2}],$$

where  $r_1 + r_2 + r_3 = m_4$ . Moreover,

$$(\partial_\kappa^{r_2}e^{-\nu_{td}\kappa^2(1+s)}) = \sum_{q_1+2q_2=r_2} C(-\nu_{td}\kappa(1+s))^{q_1}(-\nu_{td}(1+s))^{q_2}e^{-\nu_{td}\kappa^2(1+s)}.$$

As a result,

$$\begin{aligned} & \|\mathcal{R}_N(\cdot, T-s)\hat{F}(\cdot, s)\|^2 \leq \\ & \int_{\mathbb{R}} \left\| e^{-\nu_{td}\kappa^2(T-s)} \int_0^\kappa \int_0^{\kappa_N} \cdots \int_0^{\kappa_1} \partial_y^{N+1} \left[ \psi(y)e^{\Lambda_0(y)(T-s)} P_0(y)\hat{F}(y, s) \right] dy d\kappa_1 \dots d\kappa_N \right\|_Y^2 d\kappa \\ & \leq \sum_{m_1+m_2+m_3+m_4=N+1} \sup_{|\kappa| \leq 2\kappa_0} \|(\partial_\kappa^{m_1}\psi(\kappa))(\partial_\kappa^{m_3}P_0(\kappa))\|_{L(Y)}^2 \\ & \quad \times \int_{|\kappa| \leq 2\kappa_0} \left( e^{-\nu_{td}\kappa^2(T-s)} \int_0^\kappa \int_0^{\kappa_N} \cdots \int_0^{\kappa_1} \left\| \partial_y^{m_2} e^{\Lambda_0(y)(T-s)} \partial_y^{m_4} \hat{F}(y, s) \right\|_Y dy d\kappa_1 \dots d\kappa_N \right)^2 d\kappa \\ & \leq C(\psi, P_0)\nu_{td}^{2(q_1+q_2)}(T-s)^{2(\rho_1+\cdots+\rho_{m_2})}(1+s)^{\frac{2(N-1)}{3}+2(q_1+q_2)} \\ & \quad \times \int_{|\kappa| \leq 2\kappa_0} e^{-2\nu_{td}\kappa^2(T-s)} e^{-2\nu_{td}\kappa^2(1+s)} e^{2C|\kappa|^3(T-s)} |\kappa|^{2(2\rho_1+\rho_2+N+1)} |\kappa|^{2(N+1-r_1+q_1)} d\kappa \\ & \leq C(\psi, P_0)\nu_{td}^{2(q_1+q_2)}(T-s)^{2(\rho_1+\cdots+\rho_{m_2})}(1+s)^{\frac{2(N-1)}{3}+2(q_1+q_2)} \\ & \quad \times \min \left\{ (T-s)^{-(2N+2+2\rho_1+\rho_2-r_1+q_1)-\frac{1}{2}}, (1+s)^{-(2N+2+2\rho_1+\rho_2-r_1+q_1)-\frac{1}{2}} \right\}. \end{aligned}$$

As a result,

$$\begin{aligned} & \int_0^T \|\mathcal{R}_N(\cdot, T-s)\hat{F}(\cdot, s)\| ds \leq \\ & C(\psi, P_0)\nu_{td}^{(q_1+q_2)} \int_0^{T/2} (T-s)^{(\rho_1+\cdots+\rho_{m_2})}(1+s)^{\frac{(N-1)}{3}+(q_1+q_2)}(T-s)^{-(N+1+\rho_1+\rho_2/2-r_1/2+q_1/2)-\frac{1}{4}} ds \\ & + C(\psi, P_0)\nu_{td}^{(q_1+q_2)} \int_{T/2}^T (T-s)^{(\rho_1+\cdots+\rho_{m_2})}(1+s)^{\frac{(N-1)}{3}+(q_1+q_2)}(1+s)^{-(N+1+\rho_1+\rho_2/2-r_1/2+q_1/2)-\frac{1}{4}} ds \\ & \leq C(\psi, P_0)\nu_{td}^{(q_1+q_2)}(1+T)^{-\frac{N}{6}-\frac{1}{12}} \leq C(\psi, P_0, A, \chi)(1+T)^{-\frac{N}{6}-\frac{1}{12}}. \end{aligned}$$

Note that we have used the fact that

$$\begin{aligned}
& (\rho_1 + \cdots + \rho_{m_2}) + \frac{(N-1)}{3} + (q_1 + q_2) - (N+1 + \rho_1 + \rho_2/2 - r_1/2 + q_1/2) \\
&= \left( \frac{\rho_2}{2} + \rho_3 + \cdots + \rho_{m_2} \right) - \frac{2N}{3} - \frac{7}{12} + \frac{r_1}{2} + \frac{q_1}{2} + q_2 \\
&\leq \frac{1}{3}(\rho_1 + 2\rho_2 + \cdots + m_2\rho_{m_2}) - \frac{2N}{3} - \frac{7}{12} + \frac{1}{2}(r_1 + r_2) \\
&\leq \frac{1}{3}m_2 + \frac{1}{2}m_4 - \frac{2N}{3} - \frac{7}{12} \\
&\leq \frac{1}{2}(N+1) - \frac{2N}{3} - \frac{7}{12} = -\frac{N}{6} - \frac{1}{12}.
\end{aligned}$$

■

#### 4.2.4 Bounds on $\mathcal{T}_N$

In this section we show that the Taylor polynomial terms are actually zero.

**Lemma 4.20.** *Let  $\mathcal{T}_N(\kappa, T-s)\hat{G}(\kappa, s)$  be defined as in (4.26). If  $s=0$  and  $\hat{G}(\kappa, 0)$  is an initial condition for (4.8), or if  $\hat{G}(\kappa, s) = \hat{F}(\kappa, s)$ , where  $\hat{F}$  is defined in (4.6), then  $\mathcal{T}_N(\kappa, T-s)\hat{G}(\kappa, s) = 0$  for all  $\kappa$ .*

**Proof.** Recall that

$$\mathcal{T}_N(\kappa, T-s)\hat{G}(\kappa, s) = \sum_{\ell=0}^N \frac{1}{\ell!} \partial_{\kappa}^{\ell} \left( \psi(\kappa) e^{\Lambda_0(\kappa)(T-s)} P_0(\kappa) \hat{G}(\kappa, s) \right) \Big|_{\kappa=0} \kappa^{\ell}. \quad (4.28)$$

In this expression, some derivatives fall on  $\hat{G}(\kappa, s)$ , but the order of these derivatives does not exceed  $N$ . First consider the case where  $\hat{G}(\kappa, 0)$  is an initial condition for (4.8). This implies that

$$\hat{G}(\kappa, 0) = \begin{pmatrix} \hat{u}_0^s(\kappa, 0) \\ \{\hat{u}_n^s(\kappa, 0)\}_{n=1}^{\infty} \end{pmatrix}.$$

The functions  $\hat{u}_n^s$ , for  $n=0, 1, \dots$  are defined via the projections in (2.9), the similarity variables in (4.2), and the Fourier transform. Equation (2.9) defines  $w_0^s$  and  $v_n^s$  as the projections off of the first  $N+1$  eigenfunctions of the operator  $\mathcal{L}_{td}$ . The projections onto those eigenfunctions are defined in terms of the Hermite polynomials, which implies that

$$\int_{\mathbb{R}} \xi^j w_0^s(\xi, \tau) d\xi = \int_{\mathbb{R}} \xi^j v_n^s(\xi, \tau) d\xi = 0$$

for all  $\tau \geq 0$  and  $j=0, \dots, N$ . Since when  $\tau=0$  we have  $\xi=X$ , we therefore find

$$\partial_{\kappa}^j \hat{u}_0^s(\kappa, 0) \Big|_{\kappa=0} = \int_{\mathbb{R}} \mathcal{F}^{-1}[\partial_{\kappa}^j \hat{u}_0^s(\cdot, 0)](X) dX = C \int_{\mathbb{R}} X^j u_0^s(X, 0) dX = C \int_{\mathbb{R}} X^j w_0^s(X, 0) dX = 0, \quad j=0, \dots, N,$$

where we have used  $\mathcal{F}^{-1}$  to denote the inverse Fourier transform and  $C$  is some constant that can be explicitly determined. Similarly,

$$\partial_{\kappa}^j \hat{u}_n^s(\kappa, 0) \Big|_{\kappa=0} = C \int_{\mathbb{R}} X^j u_n^s(X, 0) dX = C \int_{\mathbb{R}} X^j \partial_X v_n^s(X, 0) dX = -jC \int_{\mathbb{R}} X^{j-1} v_n^s(X, 0) dX = 0, \quad j=1, \dots, N.$$

When  $j=0$ , the result holds because  $\int \partial_{\xi} v_n^s(\xi, 0) d\xi = \int V_n(\xi, 0) d\xi = 0$ , where  $V_n$  is defined in (2.4).

Next, consider the case where  $\hat{G} = \hat{F}$ . Note that  $\hat{F}(\kappa, s) = \kappa^{N+1} \hat{H}(\kappa, s)$ , where  $\hat{H}(\kappa, s)$  is a smooth, bounded function in  $\kappa$  and  $s$ . This fact can be seen from equation (4.7). Therefore  $\partial_{\kappa}^{\ell} \hat{G}(\kappa, s) \Big|_{\kappa=0} = 0$  for  $0 \leq \ell \leq N$ . ■

### 4.3 Proof of Proposition 4.2, and hence Theorem 1(ii)

Recall that the goal of this chapter is to prove Proposition 4.2, which by Remark 4.3 implies Theorem 1(ii). Hence, we want to establish the estimate

$$\|\hat{U}(\cdot, T)\| \leq C(1+T)^{-\frac{N}{6}-\frac{1}{12}}.$$

Recall from (4.8) that

$$\hat{U}(\kappa, T) = e^{\mathcal{B}(\kappa)T}\hat{U}(\kappa, 0) + \int_0^T e^{\mathcal{B}(\kappa)(T-s)}\hat{F}(\kappa, s)ds.$$

Using the splitting of the semigroup in (4.23) and Lemmas 4.16, 4.17, 4.18, 4.20, we have

$$\begin{aligned} \|\hat{U}(T)\| \leq C & \left[ e^{-\frac{\mu_1}{2}T} + e^{-MT} + (1+T)^{-\frac{N}{6}-\frac{5}{12}} \right] \|\hat{U}(0)\| \\ & + C \left[ (1+T)^{-\frac{N}{6}-\frac{1}{12}} + e^{-\frac{1}{4}MT} + (1+T)^{-\frac{N}{6}-\frac{1}{12}} \right], \end{aligned}$$

which proves the result.

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