MA 226 Worksheet
Resonance and forcing, additional spring-mass systems

1.) Consider the general damped spring-mass system

$$mx'' + bx' + kx = 0$$

a.) If $m = 3$, $b = 6$, and $k = 15$, find one solution $x(t)$, and show that the corresponding energy $E(t) = \frac{m}{2}(x')^2 + \frac{k}{2}(x)^2$ decreases in time.
b.) Show (without finding the general solution) that for a fixed spring-mass system with any positive values of m, b, and k, the energy $E(t)$ of any solution $x(t)$ decreases in time.

2.) Consider a spring mass system with mass m, damping coefficient b, and spring constant k. Suppose we are in the underdamped case (so that $b^2 - 4mk < 0$), and that ω represents the "frequency":

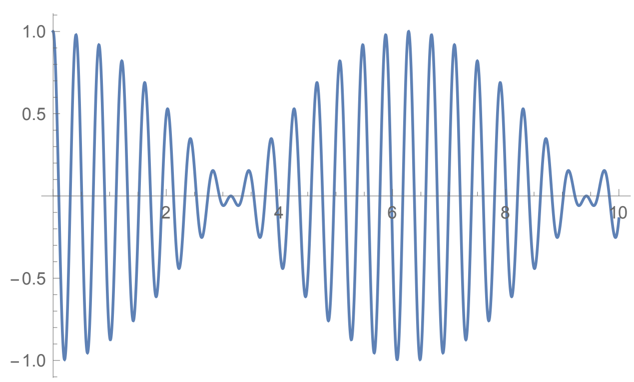
$$\omega = \frac{\sqrt{4mk - b^2}}{2m}$$

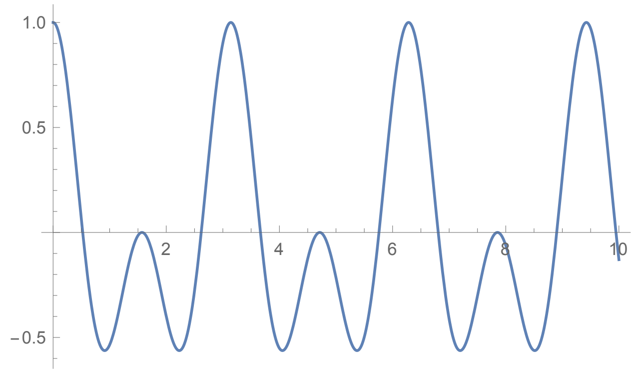
If we replace m with a larger mass, does the new system oscillate faster or slower than the original one? (Hint: think about $\frac{d\omega}{dm}$)

3.) Suppose we have four sinusoidally forced undamped spring-mass systems, with mass, spring constant, and forcing frequency as follows.

- A: $m = 2$, $k = 450$, $\omega = 16$
- B: $m = 4$, $k = 16$, $\omega = 4$
- C: $m = 1$, $k = 121$, $\omega = 11$
- D: $m = 5$, $k = 125$, $\omega = 6$

Identify which of the four systems exhibit resonance and which of the four exhibit beating, and match the graphs of solutions below with their corresponding systems.

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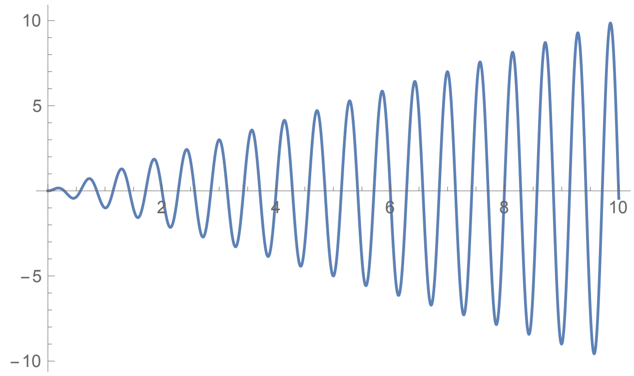
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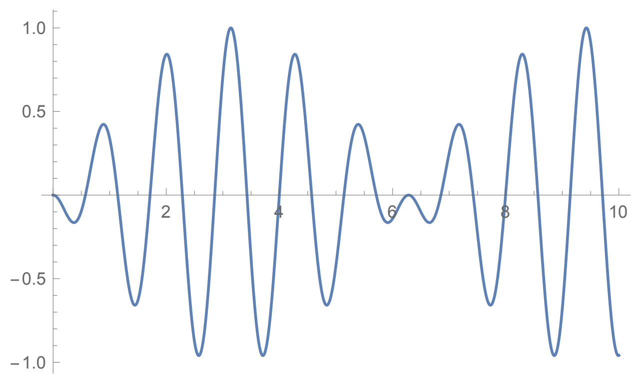
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4.) a.) Suppose an undamped system with mass m and spring constant k is externally forced by the function $f(t) = \varepsilon \cos \omega t$, where $\omega = \sqrt{k/m}$ and ε is some very small positive number. Does resonance occur in this case? What does this say about the effect of the forcing amplitude on resonance?
b.) For the system with $m = 1$, $k = 25$, and $\varepsilon = .001$, find the solution with initial position $x(0) = 0$ and $x'(0) = 1$, and determine how long it takes for the amplitude to exceed that of the corresponding unforced oscillator (i.e. the solution to the same IVP except with $\varepsilon = 0$.)

5.) Consider the forced, weakly damped spring-mass system

$$x'' + .001x' + 9x = \cos(3t)$$

Notice that in absence of the damping term, this system would exhibit resonance, but since the damping term is present, no resonance occurs. Find the solution with $x(0) = 0$ and $x'(0) = 1$, and plot it. (A computer or calculator might help). Compare the plot with the plot of the corresponding solution of the undamped system $x'' + 9x = cos(3t)$.

6.) Assume the forcing frequency ω is very close to the natural frequency ω_0. Use a Taylor expansion on the solution

$$\sin\left(\frac{(\omega_0 + \omega)t}{2}\right) \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)$$

to explain why this solution behaves like it resonates for a short time.