Prof PB's one-minute introduction to calculus

Derivatives are _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _.

Integrals are _ _ _ _.

In Calculus 1, we learn limits, derivatives, some applications of derivatives, indefinite integrals, definite integrals, and the Fundamental Theorem of Calculus.

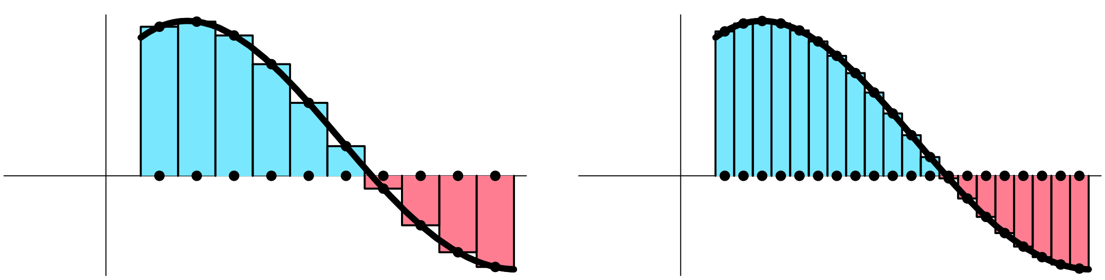
In Calculus 2, we learn some applications of the definite integral, some techniques for calculating or approximating definite integrals, infinite sequences, infinite series, and power series representations of functions.

The definition of the definite integral

For $\int_a^b f(x)\,dx$, we partition the interval $[a,b]$ into subintervals (usually of equal width).

In each subinterval $[x_{k-1}, x_k]$, we pick a "test" number $\overline{x}_k$, and we consider the Riemann sum

$$\sum_{k=1}^{n} f(\overline{x}_k)\,\Delta x.$$



If the function $f(x)$ is reasonable (for example, continuous) on the interval $[a,b]$, then the Riemann sums converge to a unique number as $n \to \infty$. That is,

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} f(\overline{x}_k)\,\Delta x\right) = \int_a^b f(x)\,dx.$$

What is the difference between $\int_a^b f(x)\,dx$ and $\int f(x)\,dx$?

Example.

$$\int_0^{3\pi/2} \sin x\,dx = \qquad\qquad \text{and} \qquad\qquad \int \sin x\,dx =$$

The Fundamental Theorem of Calculus

There are two parts to the Fundamental Theorem of Calculus. The second part gives us a method for computing integrals, and sometimes it is called the Evaluation Theorem.

Theorem. Suppose f is continuous on $[a, b]$. If F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example. We know that $-\cos x$ is an antiderivative of $\sin x$. Therefore,

$$\int_0^{3\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{3\pi/2}$$

The Method of Substitution

Example. Consider the function $\sin(x^2)$. Then

$$\frac{d}{dx}\sin(x^2) =$$

We obtain the indefinite integral

The most difficult aspect to learning the method of substitution is recognizing when it can be used. We must be able to determine if a given function is the *result* of an application of the Chain Rule. In the example above, we must realize that the function

$$2x\cos(x^2)$$

is the result of the Chain Rule applied to the function $\sin(x^2)$.

Here's how this kind of calculation is done in more abstract notation. Using the Chain Rule, we have

$$\frac{d}{dx}F(g(x)) =$$

As a result, we obtain the indefinite integral

In the example, $F(x) = \sin(x)$ and $g(x) = x^2$.

Imagine starting out trying to calculate $\int 2x\cos(x^2)\,dx$ without knowing the answer in advance. Then we would have to identify that

$$g(x) = x^2 \quad \text{and} \quad F'(x) = \cos x.$$

There is a way to simplify the procedure of identifying that an integrand has the form

$$F'(g(x))\,g'(x).$$

We refer to $g(x)$ as a change of variables, and we write $u = g(x)$. So starting with

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$$

Differential notation: We define the differential $du = \left(\frac{du}{dx}\right) dx$. Then we get

$$\int F'(u)\,du = F(u) + C.$$

This method of producing antiderivatives is called the Method of Substitution or u-substitution because it is easiest to think of the "inner function" $g(x)$ as a new variable. This new variable is often denoted by the letter u.

Example. $\int x\sqrt{9-x^2}\,dx$

You can always check your answer!