Review of The Method of Substitution

Last class we briefly talked about a method of antidifferentiation that is based on using the Chain Rule in reverse. It is called the Method of Substitution or u-substitution because it is easiest to think of the method as a “change of variables” where the “inner function” is thought of as a variable which is often denoted by the letter u.

The Chain Rule in reverse says that

$$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C.$$

We let $u = g(x)$ and we use the differential notation

$$du = \left(\frac{du}{dx}\right) dx.$$

Then we get

$$\int F'(u)\, du = F(u) + C.$$

Example. $\int x\sqrt{9-x^2}\, dx$

We let $u = 9 - x^2$. Then $du = -2x\, dx$. We obtain

$$\int x\sqrt{9-x^2}\, dx = -\frac{1}{3}\left(9-x^2\right)^{3/2} + C.$$

You can always check your answer!

Example. Match each integral on the left with one on the right and specify the substitution.

A. $\int \sin 3x \, dx$

1. $\int u^{-1/2} \, du$ with $u =$

B. $\int \frac{x-1}{x^2-2x} \, dx$

2. $\frac{1}{3} \int \sin u \, du$ with $u =$

C. $\int xe^{1-x^2} \, dx$

3. $\frac{1}{3} \int u \, du$ with $u =$

D. $\int \frac{\cos x}{\sqrt{1+\sin x}} \, dx$

4. $\int u^2 \, du$ with $u =$

E. $\int \frac{1}{\sqrt{x}(1+x)} \, dx$

5. $\frac{1}{2} \int u^{-1} \, du$ with $u =$

F. $\int (\tan^2 x)(\sec^2 x) \, dx$

6. $-\frac{1}{2} \int e^u \, du$ with $u =$

G. $\int (\sin 3x)(\cos 3x) \, dx$

7. $2 \int \frac{1}{1+u^2} \, du$ with $u =$

Substitution and definite integrals

Example. $\int_0^3 x\sqrt{9-x^2}\,dx$

We can do this definite integral in two (different) ways.

1. Calculate the indefinite integral $\int x\sqrt{9-x^2}\,dx$. Then evaluate.

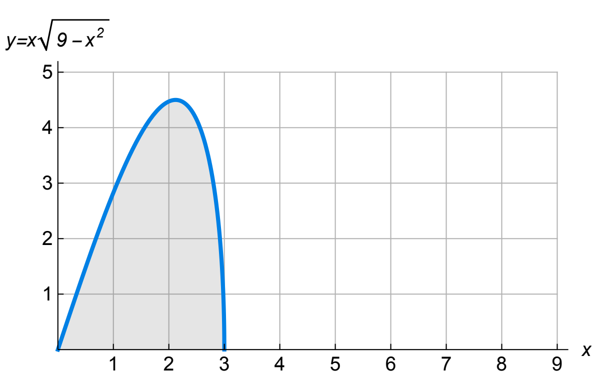
   We know that
   $$\int x\sqrt{9-x^2}\,dx = -\tfrac{1}{3}(9-x^2)^{3/2} + C.$$
   Therefore,
   $$\int_0^3 x\sqrt{9-x^2}\,dx = -\tfrac{1}{3}\left[(9-x^2)^{3/2}\right]_{x=0}^{x=3} = -\tfrac{1}{3}\Big[0-27\Big] = 9.$$

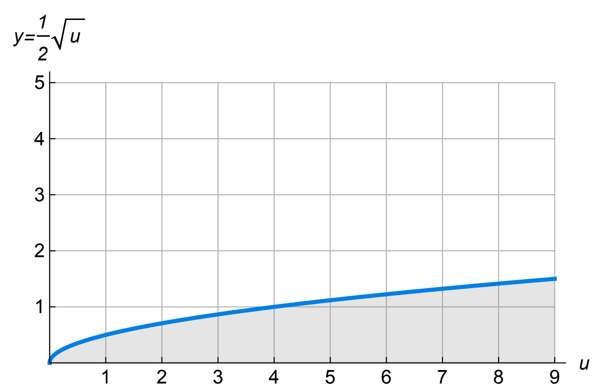
2. Change the limits of integration when you substitute.

   If $u = 9-x^2$, then $du = -2x\,dx$.

   Also, $x=0$ gives $u=9$, and $x=3$ gives $u=0$. Then
   $$\int_0^3 x\sqrt{9-x^2}\,dx = -\tfrac{1}{2}\int_9^0 \sqrt{u}\,du = \tfrac{1}{2}\int_0^9 \sqrt{u}\,du$$
   $$= \left[\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)u^{3/2}\right]_{u=0}^{u=9} = \left(\tfrac{1}{3}\right)\Big[27-0\Big] = 9.$$

The geometric interpretation of this computation is that the areas under the following two graphs are equal. The graph on the left is the graph of $f(x) = x\sqrt{9-x^2}$ and the graph on the right is the graph of $F(u) = \frac{1}{2}\sqrt{u}$.





Velocity

If we know the rate of change of a quantity, then we can calculate the value of that quantity using the definite integral and the Fundamental Theorem of Calculus.

We are most familiar with this fact when we consider motion along a line, for example, when we walk up and down Comm Ave each day. Once we pick a coordinate system, then we can describe our position by the position function $s(t)$. Then our **displacement** over the time interval $a \le t \le b$ is $s(b) - s(a)$. Displacement can be positive or negative, and we can calculate it using the Fundamental Theorem of Calculus. We get

$$\begin{aligned} \text{displacement} &= s(b) - s(a) \\ &= \int_a^b s'(t)\,dt \\ &= \int_a^b v(t)\,dt \end{aligned}$$

where $v(t) = s'(t)$ is the **velocity** of the object in question.

If we know $s(a)$ and $v(t)$ for all $a \le t \le b$, then we can obtain the position $s(b)$ by rewriting the formula for displacement as

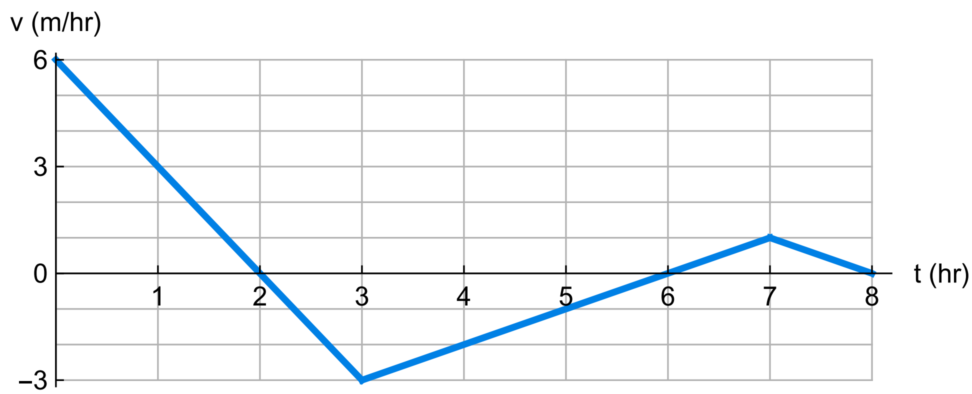
$$s(b) = s(a) + \int_a^b v(t)\,dt.$$

Note that displacement is not the same thing as distance traveled. For example, if you walk from West Campus to class and back again, you have returned to your original position, and your displacement is zero. However, your total distance traveled is definitely not zero. We calculate distance traveled by integrating $|v(t)|$. That is,

$$\text{distance traveled} = \int_a^b |v(t)|\,dt.$$

The Fundamental Theorem of Calculus tells us that the derivative of distance traveled is $|v(t)|$, and we call $|v(t)|$ the **speed** of the object.

Example. Kermit starts jogging due west on Comm Ave from Warren Towers, and the graph of his velocity is



1. What is his displacement from $t = 0$ to $t = 8$?
2. What is his distance traveled from $t = 0$ to $t = 8$?
3. Assuming that his initial position is $s(0) = 1$, what is his position at $t = 6$?