Learning Catalytics exercise: Consider the region $R$ in the $xy$-plane that is specified in the statement of the exercise in Learning Catalytics. Let $S$ be the solid that is obtained by revolving $R$ about the $y$-axis. Then the volume of $S$ is

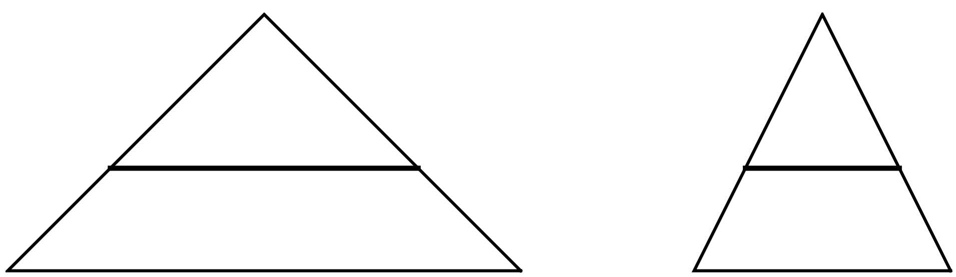
$$\int_0^1 \boxed{\phantom{integrand}}\, dy.$$

NOTE that we are integrating with respect to $y$. What integrand should we use in the integral? In other words, what expression should we use in place of the box with the word integrand in it to obtain the correct value? Since we are integrating with respect to $y$, your integrand should be a function of $y$.

A little more on the pyramid problem

**Example.** Find the volume of a pyramid that is 50 feet high with a rectangular base that is 50 feet by 100 feet.

We have identified two triangles that determine the area of any horizontal rectangular slice. In three dimensions these triangles are at right angles to one another.



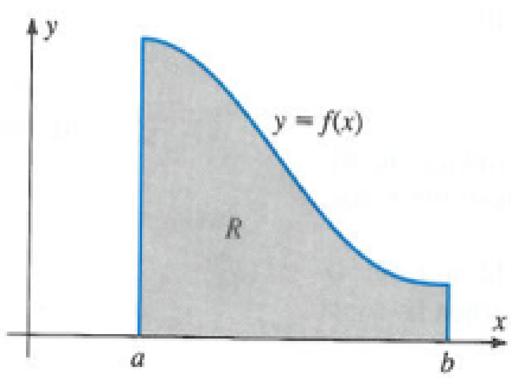
Calculating Volume Using Cylindrical Shells

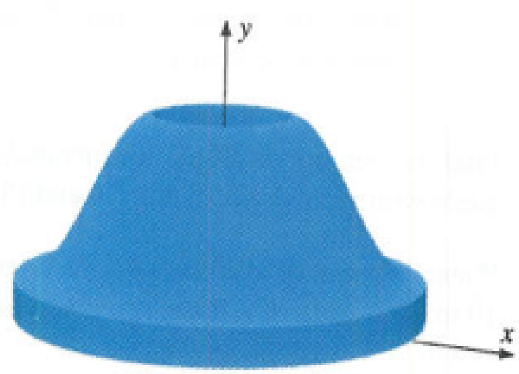
In addition to slicing methods, there is another way to calculate the volume of a solid of revolution. Slicing methods are based on the idea of taking cross sections perpendicular to the axis of rotation. The method of cylindrical shells uses hollow cylinders centered about the axis of rotation to calculate the volume.

Suppose we have a hollow cylinder of radius $r$ and height $h$. What is its area?

The method of cylindrical shells is based on this formula.

Consider the graph $y = f(x)$ of a positive function defined on the interval $a \leq x \leq b$ where both $a$ and $b$ are positive numbers. This graph determines a region $R$ in the $xy$-plane, and when we revolve $R$ about the $y$-axis, we obtain a solid of revolution.



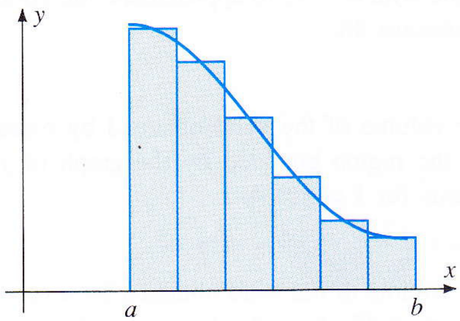



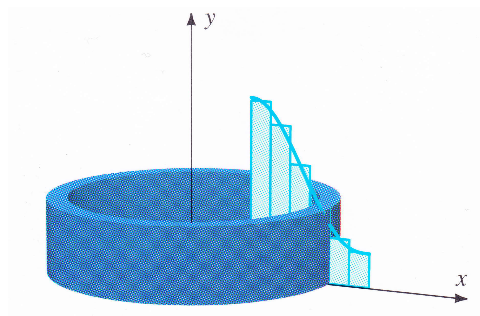

We can calculate the volume of the solid by approximating the volume with Riemann sums.

We partition the interval $a \leq x \leq b$ using equal length subintervals. We get

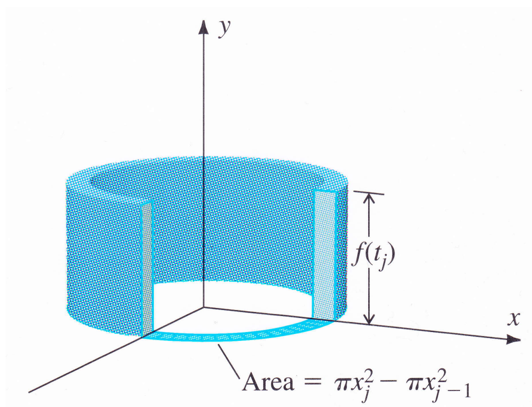
$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

where $x_j - x_{j-1} = \Delta x = (b - a)/n$. Then we choose the test numbers to be the midpoints of the subintervals. That is, $t_j = (x_{j-1} + x_j)/2$. Using these choices, we approximate the region $R$ by rectangles with the subintervals as their bases and their heights $f(t_j)$. If we rotate one of these rectangles about the $y$-axis, we obtain a cylindrical shell.



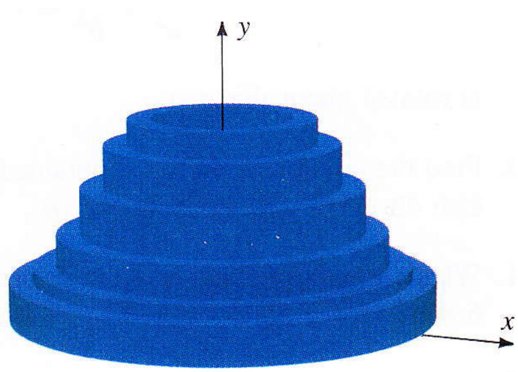

We calculate the volume of the $j$th shell.




The volume of the $j$th shell is

$$\pi(x_j^2 - x_{j-1}^2)f(t_j).$$

Combining all shells, we obtain a solid whose volume approximates the desired volume.



We have

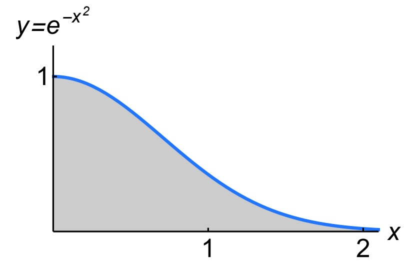
$$\text{volume} \approx \sum_{j=1}^{n} \pi(x_j^2 - x_{j-1}^2) f(t_j).$$

We want to express this sum as a Riemann sum.

We get the volume formula

$$\text{volume} = \int_a^b 2\pi x f(x)\, dx.$$

**Example.** Let $f(x) = e^{-x^2}$ over the interval $0 \le x \le 2$. What is the volume of the solid that we obtain when we rotate about the $y$-axis?




**Example.** Let $f(x) = -x^2 + 4x - 3$ and $g(x) = x - 3$. Consider the region $R$ bounded by the graphs of $y = f(x)$ and $y = g(x)$. What is the volume of the solid of revolution obtained by revolving $R$ about the $y$-axis?

We have discussed how to calculate volume using disks, washers, slices, and cylindrical shells. The disk and washer methods are just the slicing method where the slices involve one or two disks. So there are really only two methods.

Often we can calculate volume using more than one method, but usually one method is better than the others because it produces an integral that is easier to evaluate. Unfortunately, there is no straightforward way to determine the best method. A sense of which method works best on a given problem is obtained only by doing a wide range of exercises.