Learning Catalytics exercise: Consider the region R in the xy-plane that is specified in the statement of the exercise in Learning Catalytics. Let S be the solid that is obtained by revolving R about the y-axis. Then the volume of S is

$$\int_0^1 \boxed{}\, dx.$$

NOTE that we are integrating with respect to x. What integrand should we use in the integral? In other words, what expression should we use in place of the box with the word integrand in it to obtain the correct value? Since we are integrating with respect to x, your integrand should be a function of x.

More on computing volume using cylindrical shells

Last class we saw that the volume of a solid of revolution could also be obtained by decomposing the solid into cylindrical shells, and we obtained the formula

$$\text{volume} = \int_a^b 2\pi x f(x)\, dx.$$

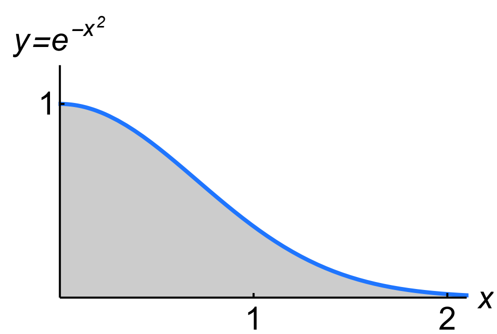
Let's return to the exercise that we did at the end of class on Friday. We can also calculate the volume using horizontal disks, that is, we integrate with respect to y.

Example. Let $f(x) = e^{-x^2}$ over the interval $0 \le x \le 2$. What is the volume of the solid that we obtain when we rotate about the y-axis?

There are two subintervals that we must consider. If $0 \le y \le e^{-4}$, then the radius of a horizontal disk is 2. If $e^{-4} < y \le 1$, then we must solve $y = e^{-x^2}$ for x. We have

$$\ln y = -x^2,$$

and therefore, $x = \sqrt{-\ln y}$. The volume is the sum of two integrals



$$\int_0^{e^{-4}} 4\pi\, dy + \int_{e^{-4}}^1 \pi(-\ln y)\, dy.$$

Compare these two integrals to the integral we obtained using cylindrical shells, that is,

$$\int_0^2 2\pi x\, e^{-x^2}\, dx.$$

Example. Let $f(x) = x$ and $g(x) = (x-2)^2$. Consider the region R bounded by the graphs of $y = f(x)$ and $y = g(x)$. What is the volume of the solid of revolution obtained by revolving R about the line $x = -1$?

We have discussed how to calculate volume using disks, washers, slices, and cylindrical shells. The disk and washer methods are just the slicing method where the slices involve one or two disks. So there are really only two methods.

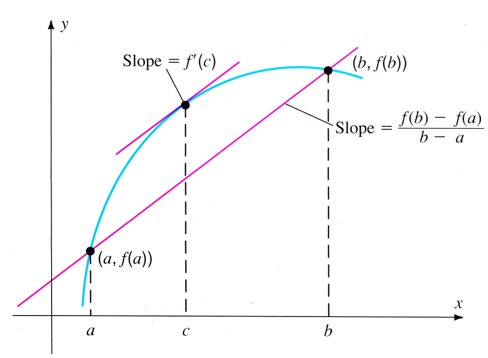
Often we can calculate volume using more than one method, but usually one method is better than the others because it produces an integral that is easier to evaluate. Unfortunately, there is no straightforward way to determine the best method. A sense of which method works best on a given problem is obtained only by doing a wide range of exercises.

Calculating Arc Length Using the Definite Integral

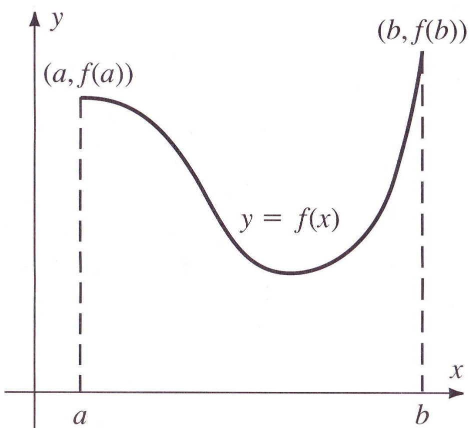
Before we begin our discussion of how the definite integral can be used to compute arc length, it is useful to recall the Mean Value Theorem.

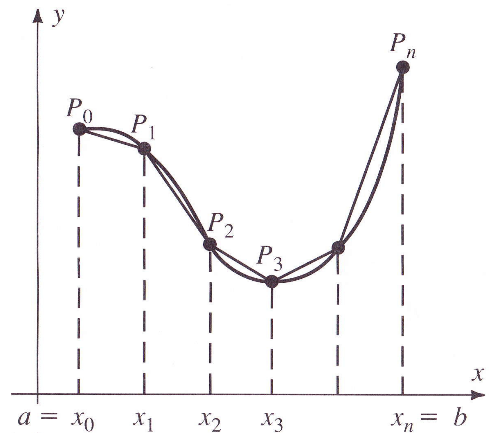
Theorem. (Mean Value Theorem) Let f be a continuous function on the interval $[a, b]$. If f is differentiable on (a, b), then there exists at least one number c between a and b for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



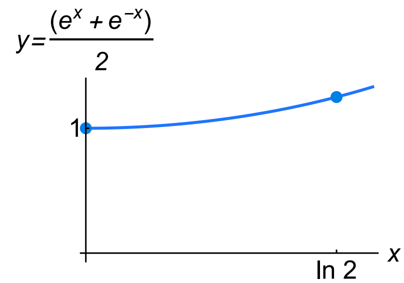
Consider the graph $y = f(x)$ of a differentiable function f. We can use the definite integral to compute its arc length.





We approximate the desired arc length by the sum $\sum_{j=1}^{n} \mathrm{dist}(P_{j-1}, P_j)$.

Example. Find the arc length of the graph of $y = \frac{1}{2}\left(e^x + e^{-x}\right)$ over the interval $0 \le x \le \ln 2$.



Sometimes it is best if we consider the curve as a function of y.

Example. Find the arc length of that portion of the curve $6xy - y^4 = 3$ that goes from $(49/48, 1/2)$ to $(14/3, 3)$.