More on Work

Here are a few more details corresponding to the ship's anchor example that we discussed at the end of the last class.

Example. Suppose that a ship's anchor weighs 2 tons (4000 pounds) in water and that the anchor is hanging taut from 100 feet of cable. How much work is required to wind in the anchor if the cable weighs 20 pounds per foot in water?

Last class we saw that the work needed to lift the cable is

$$W_c = 20 \int_0^{100} (100 - x)\ dx = 100{,}000 \text{ ft-lbs.}$$

The work needed to lift the anchor is

$$W_a = (100 \text{ ft})\,(4{,}000 \text{ lbs}) = 400{,}000 \text{ ft-lbs.}$$

So the total work needed to lift the anchor and its cable is $W_a + W_c = 500{,}000$ ft-lbs.

Example. A water tank has the shape of an inverted circular cone with a height of 5 m and a base radius of 2 m. It is filled with water to a height of 4 m. How much work is required to empty the tank by pumping all of the water to the top of the tank?

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Exponential Models

Suppose that $y(t)$ is a quantity that varies with time t. Then dy/dt is its rate of change. However, it is often common to refer to rates of change in a relative fashion. That is,

$$\frac{dy/dt}{y}.$$

For example, the gross domestic product of China grew at an annual rate of 10% in 2010. Quantities that grow at a constant relative growth rate grow in an exponential fashion. Similarly, quantities that decay at a constant relative decay rate decay exponentially.

If the relative rate of change of a quantity $y(t)$ is a constant k, then

$$\left(\frac{1}{y}\right)\frac{dy}{dt} = k,$$

so $y(t)$ satisfies the differential equation $\dfrac{dy}{dt} = ky$.

Theorem. If the function $y(t)$ satisfies the differential equation $\dfrac{dy}{dt} = ky$, then $y(t) = y_0 e^{kt}$ where $y_0 = y(0)$.

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Example. How long does it take a supply of radium to decay to 90% of its original amount? (Note: The half-life of radium is 1600 years.)