Learning Catalytics exercise: Here's some space for you to do a quick calculation.

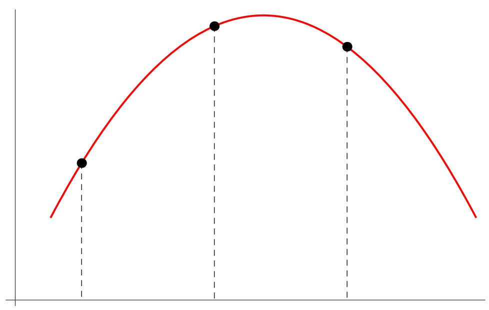
Simpson's Rule

Both the Midpoint Rule and the Trapezoid Rule are based on the use of line segments to approximate the graph of the given function f. If the graph of f is curved, it often makes more sense to approximate the graph (and consequently the integral) using well-known curves. Simpson's Rule uses parabolic arcs. The procedure is based on the fact that, if x_0, x_1, and x_2 are three numbers such that

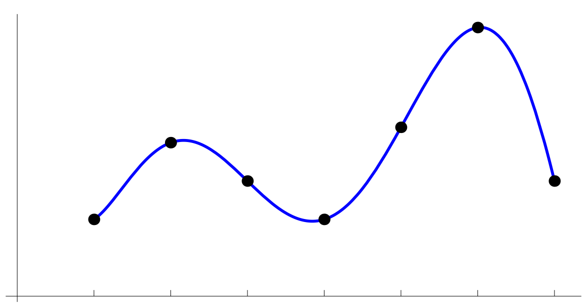
$$x_1 - x_0 = x_2 - x_1 = \Delta x,$$

then the definite integral of the parabola passing through the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) equals

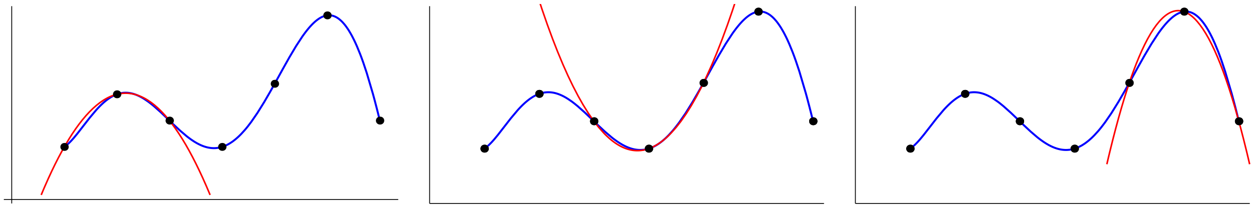
$$\frac{\Delta x}{3}\left[y_0 + 4y_1 + y_2\right].$$



Simpson's Rule only applies to the case where there are an even number n of subdivisions.



We use the first three points to form a parabola. Then we use the third, fourth, and fifth points to form a second parabola, etc.



Simpson's Rule

Example. Approximate $\int_0^6 \sqrt{1+x^4}\,dx$ using Simpson's Rule with $n = 6$.

Remark. One can show that $S(2n) = \frac{2}{3}M(n) + \frac{1}{3}T(n)$.

Error estimates

Remark. Typically the Midpoint Rule is twice as accurate as the Trapezoid Rule.

There are formal error estimates for all three rules. Here is the error estimate for the Midpoint and the Trapezoid Rules.

Theorem. Assume that f'' is continuous on the interval $[a, b]$ and that k is a bound on the absolute value of the second derivative of f on $[a, b]$. That is, $|f''(x)| \leq k$ for all x in $[a, b]$. Then we have

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \quad \text{and} \quad E_T \leq \frac{k(b-a)}{12}(\Delta x)^2$$

where E_M is the error involved in estimating the integral with the Midpoint Rule and E_T is the error involved in estimating the integral with the Trapezoid Rule.

Here is the error estimate for Simpson's Rule.

Theorem. Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a bound on the absolute value of the fourth derivative of f on $[a, b]$. That is, $|f^{(4)}(x)| \leq K$ for all x in $[a, b]$. Then we have

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4$$

where E_S is the error involved in estimating the integral with Simpson's Rule.

Example. Estimate $\int_0^1 \cos(x^2)\, dx$ using the Trapezoid Rule $T(4)$ and bound the error involved in this estimate.

Because

$$\Delta x = \frac{b-a}{n},$$

the error estimates can also be written as

$$E_M \leq \frac{k(b-a)^3}{24\,n^2}, \quad E_T \leq \frac{k(b-a)^3}{12\,n^2}, \quad \text{and} \quad E_S \leq \frac{K(b-a)^5}{180\,n^4}$$