More on improper integrals

At the end of last class, we were in the middle of discussing the following very important family of examples.

**Example.** Consider the family of improper integrals \( \int_1^{\infty} \frac{1}{x^p} \, dx \) where \( p \) is any real constant.
Now we consider improper integrals where the function is unbounded on the interval of integration. However, we start with a calculation that is wrong.

**Example.** \( \int_{-1}^{1} \frac{1}{x^2} \, dx \)

![Graph of the function \( \frac{1}{x^2} \)](image)

**Definition.** Suppose that the function \( f \) is continuous on the half-open interval \((a, b]\) with

\[
\lim_{x \to a^+} f(x) = \pm \infty.
\]

Then

\[
\int_{a}^{b} f(x) \, dx = \lim_{a \to a^+} \int_{a}^{b} f(x) \, dx,
\]

provided the limit exists and is finite.

In general, if the function \( f \) is continuous on the interval \([a, b]\) except at some number \( p \) between \( a \) and \( b \) and if the graph of \( f \) has a vertical asymptote at \( p \), then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{p} f(x) \, dx + \int_{p}^{b} f(x) \, dx,
\]

provided that both improper integrals on the right-hand side exist.
Example. Find the arc length of the semicircle that is the graph of \( y = \sqrt{1 - x^2} \) for \(-1 \leq x \leq 1\).
Example. $\int_{-1}^{1} \frac{1}{x^2} \, dx$

Example. $\int_{0}^{1} \frac{x}{\sqrt{1 - x^2}} \, dx$