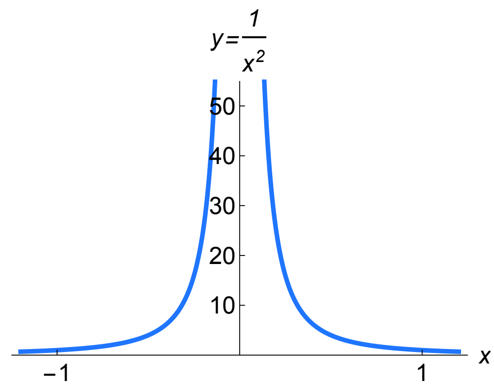
Improper integrals for functions with vertical asymptotes

We consider improper integrals where the function is unbounded on the interval of integration. However, we start with a calculation that is wrong.

Example. $\int_{-1}^{1} \frac{1}{x^2}\, dx$



Definition. Suppose that the function f is continuous on the half-open interval $(a, b]$ with

$$\lim_{x\to a^+} f(x) = \pm\infty.$$

Then

$$\int_a^b f(x)\, dx = \lim_{\alpha\to a^+} \int_\alpha^b f(x)\, dx,$$

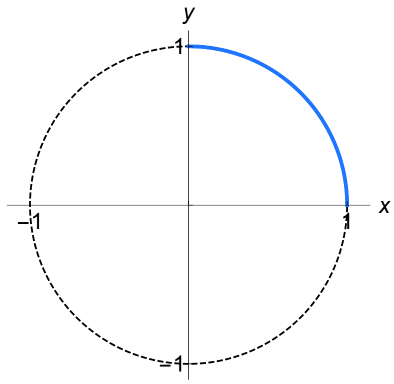
provided the limit exists and is finite.

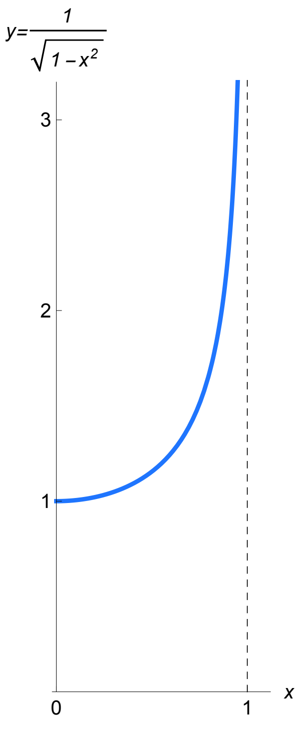
In general, if the function f is continuous on the interval $[a, b]$ except at some number p between a and b and if the graph of f has a vertical asymptote at p, then

$$\int_a^b f(x)\, dx = \int_a^p f(x)\, dx + \int_p^b f(x)\, dx,$$

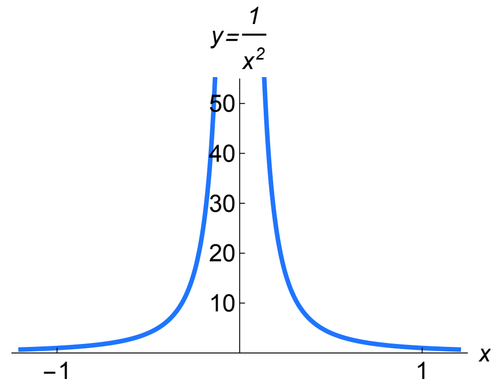
provided that both improper integrals on the right-hand side exist.

Example. Find the arc length of the quarter circle that is the graph of $y = \sqrt{1 - x^2}$ for $0 \le x \le 1$.





Example. $\int_{-1}^{1} \frac{1}{x^2}\,dx$



Example. $\int_{0}^{1} \frac{x}{\sqrt{1-x^2}}\,dx$