

Infinite sequences

For most of the remainder of the semester, we are going to study infinite sequences, infinite series, and power series. These topics were developed so that we can make sense of expressions such as

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

We start with infinite sequences. Roughly speaking an infinite sequence is just an infinitely long list of numbers. For example,

$$1, 4, 7, 10, \dots$$

is an infinite sequence of numbers. More precisely, we have the following definition.

Definition. An infinite sequence is a function whose domain is the set of positive (or nonnegative) integers.

Example. The sequence

$$1, 4, 7, 10, \dots$$

is given by the function

$$f(n) = 3n - 2$$

for $n = 1, 2, 3, \dots$

We usually use subscript notation rather than function notation when we talk about sequences. For the above sequence, we write

$$a_n = 3n - 2.$$

Here are more examples of sequences:

Example. $1, 2, 4, 8, \dots$

Example. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

Example. The Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...

Example. 1, 11, 21, 1211, 111221, 312211, ...

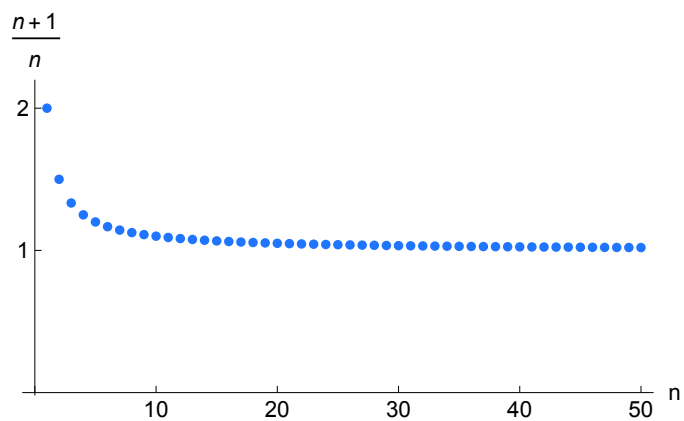
In this course, we will mainly be concerned with the limit of a sequence as $n \rightarrow \infty$.

“**Definition.**” We say that

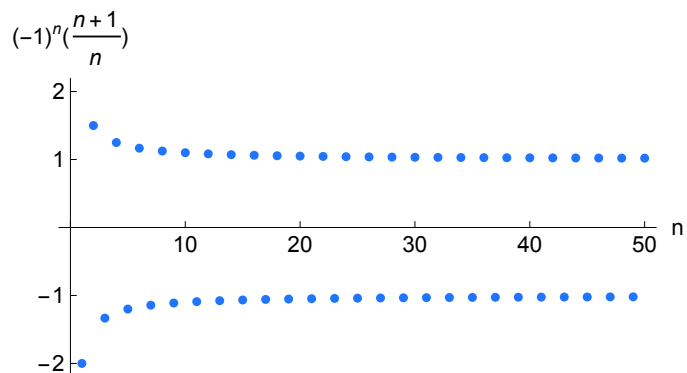
$$\lim_{n \rightarrow \infty} a_n = L$$

if the numbers a_n approach the number L as n increases without bound.

Example. Consider the statement $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$.



Example. What about $\lim_{n \rightarrow \infty} (-1)^n \frac{n+1}{n}$?



How do we compute limits of sequences?

First, the standard limit theorems are the same for sequences as they are for functions. For example, the limit of the sum of two sequences is the sum of the limits, etc.

Definition. Given a real number r , the geometric sequence with ratio r is the sequence $a_n = r^n$.

Theorem.

- If $|r| < 1$, then the limit of r^n as $n \rightarrow \infty$ is 0.
- If $r > 1$, then the limit of r^n as $n \rightarrow \infty$ is ∞ .
- If $r = 1$, then the limit of r^n as $n \rightarrow \infty$ is 1.
- If $r \leq -1$, then the limit of r^n as $n \rightarrow \infty$ does not exist.

Theorem. Suppose that $a_n = f(n)$ for some function f defined on the interval $[1, \infty)$. If

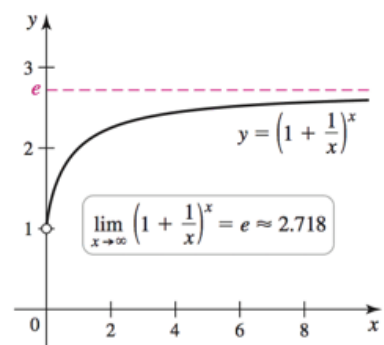
$$\lim_{x \rightarrow \infty} f(x) = L,$$

then

$$\lim_{n \rightarrow \infty} a_n = L.$$

Example. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)$

Example. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$



Example. $\lim_{n \rightarrow \infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$