Infinite sequences and series—a brief review

An **infinite sequence** is an infinite list of numbers. That is, an infinite sequence is $a_1,\ a_2,\ a_3,\ldots$. In this course we are mainly concerned with the limit

$$\lim_{n\to\infty} a_n.$$

An **infinite series** is the sum of an infinite list of numbers. That is, an infinite series is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots .$$

We consider the sequence of **partial sums**. That is, we define its sequence of partial sums $\{S_n\}$ by

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \end{aligned}$$

Learning Catalytics exercise: Here's some space in case you need to do a quick calculation.

Definition. The infinite series $a_1 + a_2 + a_3 + \ldots$ **converges** if the limit

$$\lim_{n\to\infty} S_n$$

exists and is finite. Otherwise, the infinite series **diverges.**

Geometric series

Definition. A geometric series is one in which the ratio of successive terms is constant. In other words, there is a number r such that

$$\frac{a_{k+1}}{a_k} = r \text{ for all } k.$$

Theorem. Consider the geometric series $a + ar + ar^2 + \dots$ where $a \neq 0$.

- If $|r| < 1$, then the series converges to $\frac{a}{1-r}$.
- If $|r| \geq 1$, then the series diverges.

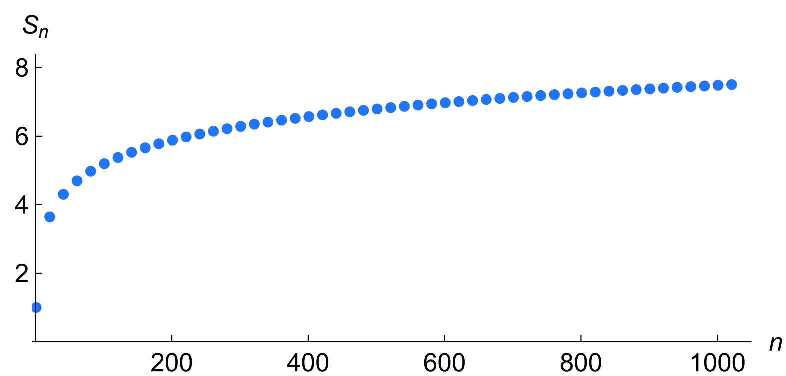
We can also determine the sum of a series if it is a telescoping series.

Sometimes it is very difficult to tell if a series converges by looking at a graph of its partial sums.

Example. The harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Here is a graph of its nth partial sums.



Does the harmonic series converge?

Tests for convergence

Most infinite series do not yield explicit expressions for their nth partial sums S_n. Therefore, we concentrate on "tests for convergence" that do not require that we determine a formula for S_n.

The most basic of these tests is the Divergence Test (sometimes called the kth Term Test). Suppose that the infinite series

$$\sum_{k=1}^{\infty} a_k$$

converges.

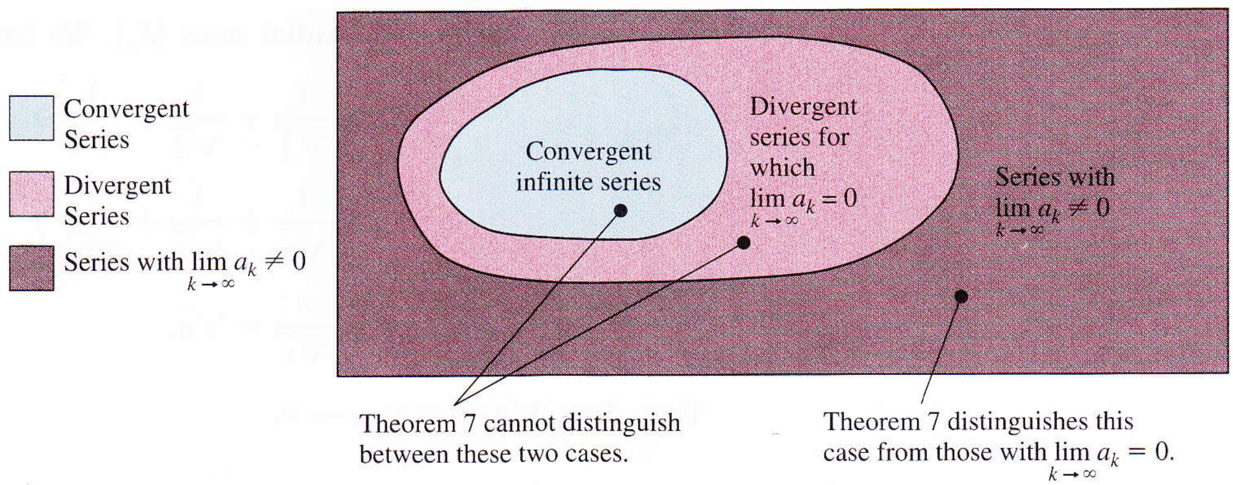
Theorem. If the infinite series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

The kth Term Test for Divergence: If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Example. $\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$

Example. $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (the harmonic series)

Remark. It is important to remember the harmonic series when one thinks about how the kth term test is used. *The kth term test never establishes convergence of a series.* It can only be used to conclude that a series diverges.



Convergence tests for positive term series

Suppose that we have a series $a_1 + a_2 + a_3 + \dots$ where all of the terms a_k are positive. Then the sequence of partial sums S_n is a monotonically increasing sequence. That is, $S_1 < S_2 < S_3 < \dots$.

Theorem. A positive term series converges if and only if its sequence of partial sums is bounded above.

Example. Note that this theorem does not hold for arbitrary series. For example, the partial sums of $1 - 1 + 1 - 1 + 1 + \dots$ are bounded above by $M = 1$. Nevertheless, this series diverges.

This theorem gives us a strategy for determining the convergence of a series with positive terms. We look for ways to bound the sequence of partial sums.

The Integral Test

One way to bound the sequence of partial sums is to use improper integrals. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots .$$