More on the Integral Test

Last class we saw two examples of how improper integrals can be used to determine if a series converges or diverges. The estimates that we used in those examples can be generalized to obtain the Integral Test.

**Theorem.** (Integral Test) Let $a_k = f(k)$ for $k = 1, 2, 3, \ldots$ where $f$ is a function that is continuous, positive, and decreasing on the interval $[1, \infty)$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_{1}^{\infty} f(x) \, dx$$

either both converge or both diverge. If they converge, the value of the integral is not, in general, the value of the series.

**Example.** \[ \sum_{k=1}^{\infty} \frac{k}{k^2 + 1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \ldots \]
$p$-series

There is an important class of series, the $p$-series, whose convergence can be determined by the Integral Test.

**Definition.** Given a real number $p$, the $p$-series is the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$.

**Examples.** The following two series are $p$-series.

1. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$

2. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots$

Before we apply the Integral Test to these series, we need to recall the family of improper integrals

$$\int_{1}^{\infty} \frac{1}{x^p} \, dx$$

where $p$ is any real constant.
On February 25, we calculated that

\[
\int_1^\infty \frac{1}{x^p} \, dx = \begin{cases} 
\text{converges to } & \frac{1}{p-1} \text{ if } p > 1 \\
\text{diverges} & \text{if } p \leq 1.
\end{cases}
\]

Therefore, we can use the Integral Test to determine the convergence of all \(p\)-series.

**Theorem.** Consider the \(p\)-series \(\sum_{k=1}^\infty \frac{1}{k^p}\).

- It converges if \(p > 1\), and
- it diverges if \(p \leq 1\).

**Examples.** Consider the two series

1. \(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots\)

2. \(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots\)

Which of these two series converge?
Approximating infinite series using the Integral Test

Suppose that the Integral Test applies to

\[ \sum_{k=1}^{\infty} a_k \]

where \( a_k = f(k) \) for some decreasing function \( f \). Recall that the \( n \)th partial sum is

\[ S_n = a_1 + a_2 + \ldots + a_n. \]

We can use improper integrals to approximate the infinite sum.

**Theorem.**

\[ S_n + \int_{n+1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_{n}^{\infty} f(x) \, dx \]

To see why, we sketch two figures:
Example. Estimate the accuracy of the approximation \[ \sum_{k=1}^{\infty} \frac{1}{k^2} \approx \sum_{k=1}^{100} \frac{1}{k^2}. \]

Example. How many terms of the series \( \sum_{k=1}^{\infty} \frac{1}{k^3} \) must be summed to estimate the value of the series within an error of \( 10^{-3} \)?