Learning Catalytics exercise: Here's some space in case you need to do a quick calculation.

Convergence tests for positive term series

Suppose that we have a series $a_1 + a_2 + a_3 + \dots$ where all of the terms a_k are positive. Then the sequence of partial sums S_n is a monotonically increasing sequence. That is, $S_1 < S_2 < S_3 < \dots$.

Theorem. A positive term series converges if and only if its sequence of partial sums is bounded above.

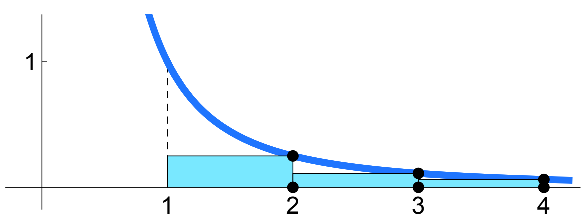
Example. Note that this theorem does not hold for arbitrary series. For example, the partial sums of $1 - 1 + 1 - 1 + 1 + \dots$ are bounded above by $M = 1$. Nevertheless, this series diverges.

This theorem gives us a strategy for determining the convergence of a series with positive terms. We look for ways to bound the sequence of partial sums.

The Integral Test

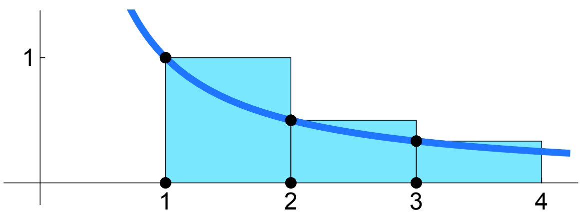
One way to bound the sequence of partial sums is to use improper integrals. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots .$$



Now consider the harmonic series

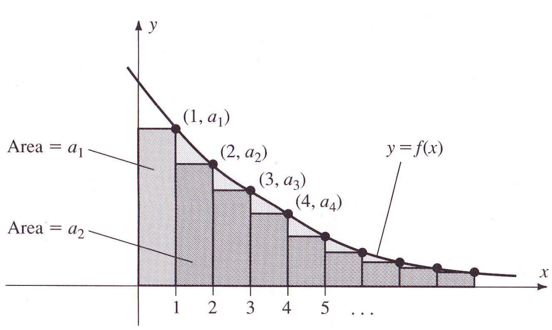
$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots .$$

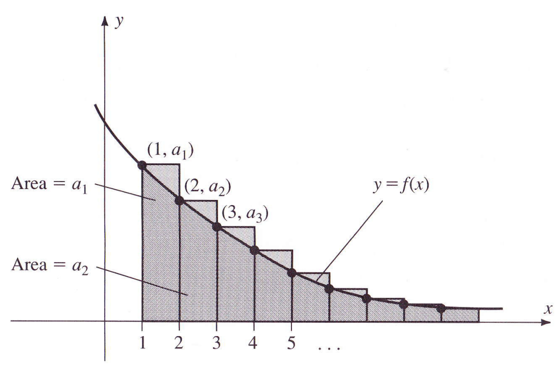


Theorem. (Integral Test) Let $a_k = f(k)$ for $k = 1, 2, 3, \dots$ where f is a function that is continuous, positive, and decreasing on the interval $[1, \infty)$. Then

$$\sum_{k=1}^{\infty} a_k \qquad \text{and} \qquad \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge. If they converge, the value of the integral is not, in general, the value of the series.





Example. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \dots$

p-series

There is an important class of series, the p-series, whose convergence can be determined by the Integral Test.

Definition. Given a real number p, the p-series is the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$.

Examples. The following two series are p-series.

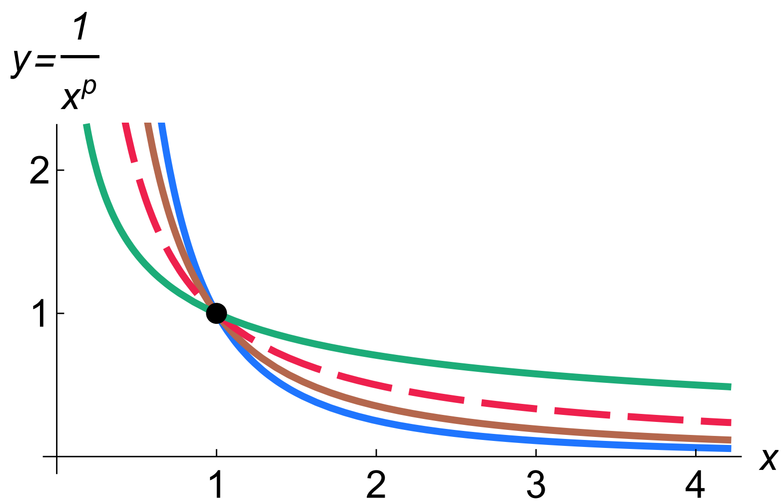
1. $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \dots$

2. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dots$

Before we apply the Integral Test to these series, we need to recall the family of improper integrals

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

where p is any real constant.



On February 22, we calculated that

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{converges to } \dfrac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Therefore, we can use the Integral Test to determine the convergence of all p-series.

Theorem. Consider the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$.

- It converges if $p > 1$, and
- it diverges if $p \leq 1$.

Examples. Consider the two series

1. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$

2. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$

Which of these two series converge?