More on approximating infinite series using the Integral Test

Suppose that the Integral Test applies to the convergent infinite series

$$\sum_{k=1}^{\infty} a_k$$

where $a_k = f(k)$ for some decreasing function $f$. We can use partial sums and improper integrals to approximate the infinite sum. Recall that the $n$th partial sum is

$$S_n = a_1 + a_2 + \ldots + a_n.$$  

Define the infinite series

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots.$$ 

Note that $R_n$ converges and

$$S_n + R_n = \sum_{k=1}^{\infty} a_k.$$ 

**Theorem.** The series $R_n$ satisfies the inequality

$$\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx.$$ 

Consequently,

$$S_n + \int_{n+1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_{n}^{\infty} f(x) \, dx.$$
Example. Estimate the accuracy of the approximation \( \sum_{k=1}^{\infty} \frac{1}{k^2} \approx \sum_{k=1}^{100} \frac{1}{k^2} \).
Example. How many terms of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ must be summed to estimate the value of the series within an error of $10^{-3}$?
The Comparison Test

Consider the two series

$$
\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots
$$

and

$$
\sum_{k=1}^{\infty} \frac{k}{k^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \ldots.
$$

Here is a graph of their partial sums:

![Graph of partial sums]

**Theorem.** (Comparison Test) Given two series

$$
\sum a_k \quad \text{and} \quad \sum b_k
$$

where the terms $a_k$ and $b_k$ are positive. Then

1. If $\sum b_k$ converges and $a_k < b_k$ for all $k$, then $\sum a_k$ converges.

2. If $\sum a_k$ diverges and $a_k < b_k$ for all $k$, then $\sum b_k$ diverges.
Example. \( \sum_{k=1}^{\infty} \frac{1}{2k-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \)

Example. \( \sum_{k=1}^{\infty} \frac{\sqrt{k} - 1}{k^2 + 2} \)