Approximating infinite series using the Integral Test

Suppose that the Integral Test applies to the convergent infinite series

$$\sum_{k=1}^{\infty} a_k$$

where $a_k = f(k)$ for some positive, decreasing function f. We can use partial sums and improper integrals to approximate the infinite sum. Recall that the nth partial sum is

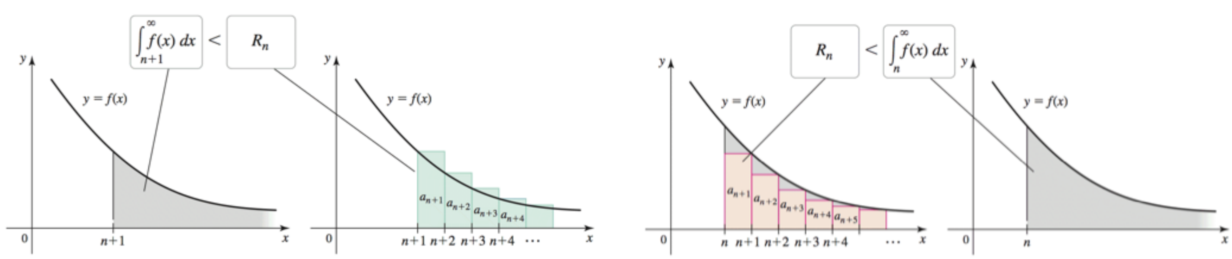
$$S_n = a_1 + a_2 + \ldots + a_n.$$

Define the infinite series

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots .$$

Note that R_n converges and

$$S_n + R_n = \sum_{k=1}^{\infty} a_k.$$



Theorem. The series R_n satisfies the inequality

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Consequently,

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_{n}^{\infty} f(x)\,dx.$$

Example. How many terms of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ must be summed to estimate the value of the series within an error of 10^{-3}?

We improve the estimate using L_n and U_n.

The Comparison Test

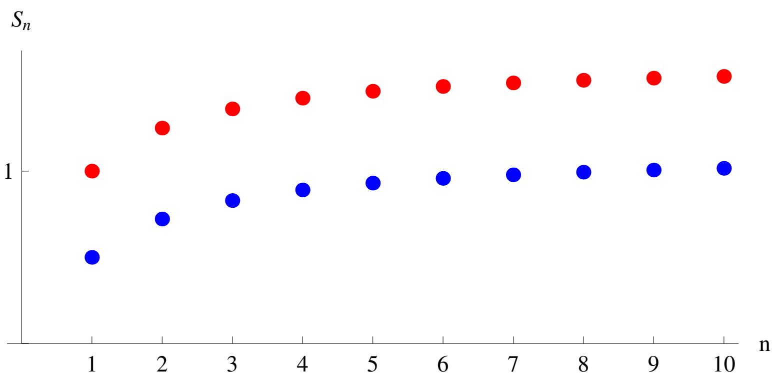
Consider the two series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

and

$$\sum_{k=1}^{\infty} \frac{k}{k^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \dots .$$

Here is a graph of their partial sums:



Theorem. (Comparison Test) Given two series

$$\sum a_k \qquad \text{and} \qquad \sum b_k$$

where the terms a_k and b_k are positive. Then

1. If $\sum b_k$ converges and $a_k < b_k$ for all k, then $\sum a_k$ converges.
2. If $\sum a_k$ diverges and $a_k < b_k$ for all k, then $\sum b_k$ diverges.

Example. $\sum_{k=2}^{\infty} \frac{k^3}{k^4-1} = \frac{8}{15} + \frac{27}{80} + \frac{64}{255} + \frac{125}{624} + \dots$

Example. $\sum_{k=2}^{\infty} \frac{\sqrt{k-1}}{k^2+2} = \frac{1}{6} + \frac{\sqrt{2}}{11} + \frac{\sqrt{3}}{18} + \frac{2}{27} + \dots$