Limit comparison test

Example. \[ \sum_{k=1}^{\infty} \frac{\sqrt{k}}{1+k} = \frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} + \ldots \]

Do you think that this series converges?
Theorem. (Limit Comparison Test) Suppose that \( \sum a_k \) and \( \sum b_k \) are two series with positive terms and
\[
\lim_{k \to \infty} \frac{a_k}{b_k} = L.
\]

1. If \( 0 < L < \infty \), then \( \sum a_k \) and \( \sum b_k \) either both converge or both diverge.

2. If \( L = 0 \) and \( \sum b_k \) converges, then \( \sum a_k \) converges.

3. If \( L = \infty \) and \( \sum b_k \) diverges, then \( \sum a_k \) diverges.

Let’s consider the preceding example using the Limit Comparison Test.

Example. \( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{1+k} = \frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} + \ldots \)

Why is the Limit Comparison Test true?
Example. \( \sum_{k=3}^{\infty} \frac{1}{\sqrt{k^3 - k^2 - 2k}} = \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{40}} + \frac{1}{\sqrt{90}} + \ldots \)

Example. \( \sum_{k=2}^{\infty} \frac{\ln k}{k^2} = \frac{\ln 2}{4} + \frac{\ln 3}{9} + \frac{\ln 4}{16} + \ldots \)
The Ratio Test

The Ratio Test takes advantage of what we know about geometric series and the Comparison Test.

**Example.** \( \sum_{k=1}^{\infty} \frac{k + 1}{k!} = 2 + \frac{3}{2} + \frac{2}{3} + \frac{5}{24} + \ldots \)
Theorem. (Ratio Test) Let \( \sum a_k \) be a series with positive terms and let

\[
r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}.
\]

1. If \( 0 \leq r < 1 \), then \( \sum a_k \) converges.
2. If \( r > 1 \), then \( \sum a_k \) diverges.
3. If \( r = 1 \), the test is inconclusive.