Alternating series

An alternating series is one in which the terms alternate between positive and negative numbers.

**Example.** \(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \ldots\)

Here is a plot of its partial sums:
Theorem 1. Consider the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

with $a_k > 0$. It converges if the following two conditions hold:

1. The $a_k$ satisfy the condition that $a_{k+1} \leq a_k$ for all $k$.
2. $\lim_{k\to\infty} a_k = 0$.

If so, the series converges to a value $S$ between 0 and $a_1$.

Example. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 + 4} = \frac{1}{5} - \frac{1}{4} + \frac{3}{13} - \frac{1}{5} \pm \ldots$

Here is a plot of its partial sums:

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Estimating the remainder in an alternating series

The $n$th remainder $R_n$ for a series that converges to $S$ is $R_n = |S - S_n|$.

**Theorem 2.** For an alternating series that satisfies the hypothesis of Theorem 1, then

$$R_n < a_{n+1}.$$
Example. How large must $n$ be so that the $n$-th partial sum $S_n$ approximates

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \ldots$$

with a remainder that is less than 0.05?

Example. How many terms of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

do we need to sum to be sure that the remainder is less than $10^{-4}$? (We shall see that this infinite sum converges to $1/e$.)

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