Better figure to illustrate the computation we made at the end of last class

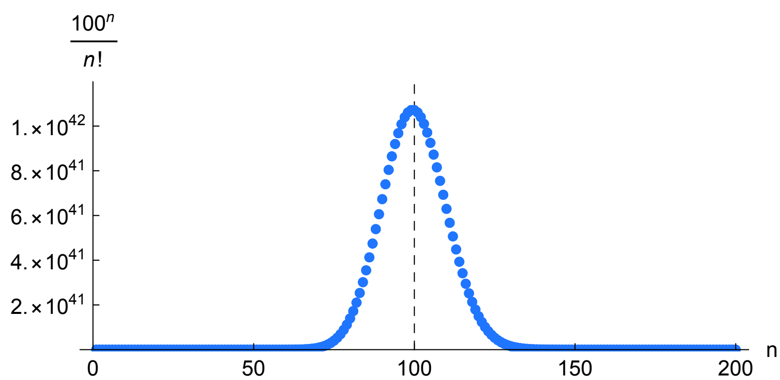
Last class we compared $b^n$ and $n!$ in the case where $b = 100$. We ended up with some impressive numbers such as

$$\frac{100^{30}}{30!} = \frac{19073486328125000000000000000000000000000}{5059296795123883412 1} \approx 3.77 \times 10^{27}.$$

If we plot the ratio

$$\frac{100^n}{n!}$$

for $1 \leq n \leq 200$, we get the following figure. Note that the axes do not cross at $(0, 0)$. The second coordinate is considerably greater than 0.




The figure suggests that $100^n << n!$. Here is how we can verify this fact:

Let

$$M = \frac{100^{100}}{100!} \approx 1.07 \times 10^{42}.$$

We will see that $M$ is the largest number in this infinite sequence. In other words, $M$ is the maximum value in the graph above.

Suppose that $n$ is any integer that is greater than 100. We write

$$\frac{100^n}{n!} = \left(\frac{100}{1}\right)\left(\frac{100}{2}\right)\cdots\left(\frac{100}{100}\right)\left(\frac{100}{101}\right)\left(\frac{100}{102}\right)\cdots\left(\frac{100}{n}\right).$$

We group the first 100 factors together and get $M$, that is,

$$M = \left(\frac{100}{1}\right)\left(\frac{100}{2}\right)\cdots\left(\frac{100}{100}\right).$$

Therefore,

$$\frac{100^n}{n!} = M\left(\frac{100}{101}\right)\left(\frac{100}{102}\right)\cdots\left(\frac{100}{n}\right).$$

The number $M$ is a (large) constant that does not depend on $n$. Next we observe that

$$\frac{100^n}{n!} = M\left(\frac{100}{101}\right)\left(\frac{100}{102}\right)\cdots\left(\frac{100}{n}\right) < M\left(\frac{100}{101}\right)^{n-100}.$$

Then

$$\lim_{n\to\infty}\frac{100^n}{n!} \leq M\lim_{n\to\infty}\left(\frac{100}{101}\right)^{n-100} = (M)(0) = 0$$

because

$$\left\{\left(\frac{100}{101}\right)^{n-100}\right\}$$

is a geometric sequence with ratio $r = 100/101$ and $|r| < 1$ in this case.

This argument shows that $100^n << n!$.

There is nothing special about the base 100 in this argument. How would you modify the argument to handle other bases $b > 1$ such as $b = \pi$, $b = 10^{100}$ (a googol), or $b = 100!$ ?

We conclude that $b^n << n!$ for any $b > 1$.