More on infinite sequences

Last class we discussed ways for computing

$$\lim_{n\to\infty} a_n$$

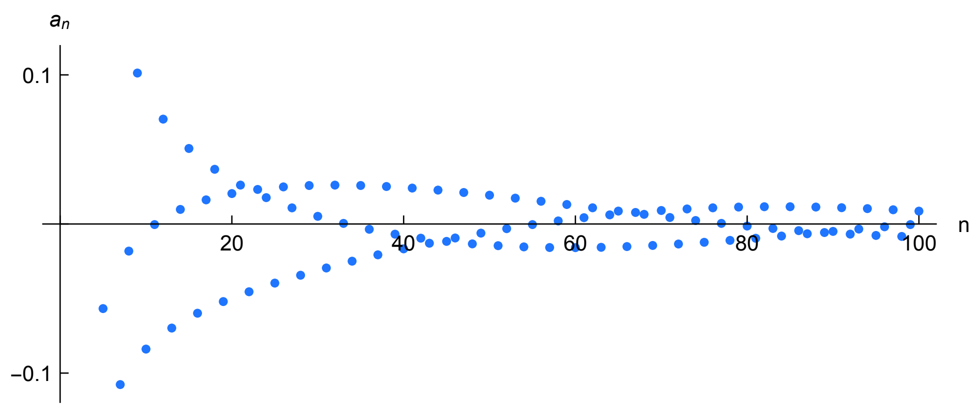
for various sequences a_n. In particular, we discussed geometric sequences and sequences that arise as restrictions of functions that are defined on intevals such as the interval $1 \leq x < \infty$. Today we continue to discuss ways in which we can calculate limits.

Theorem. (Squeeze Theorem) If a_n, b_n, and c_n are three sequences such that

$$a_n \leq b_n \leq c_n$$

for all n and if $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

Example. $\lim\limits_{n\to\infty} (-1)^n \dfrac{\cos n}{n}$



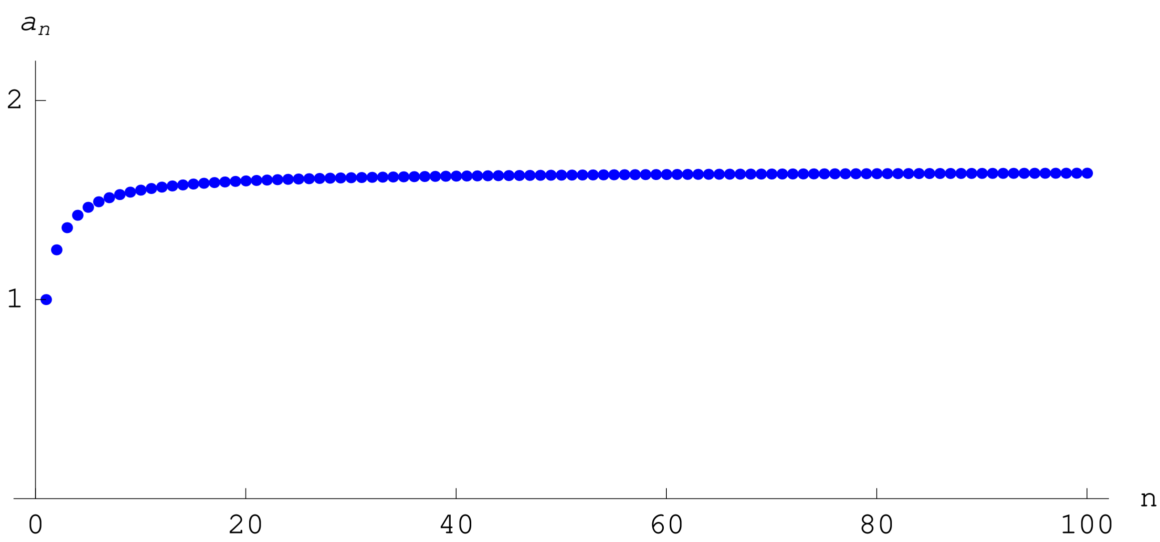
Bounded monotonic sequences

Definition. A sequence is **monotonically increasing** if $a_{n+1} \geq a_n$ for all n.

Example. The sequence a_n defined recursively by $a_1 = 1$ and

$$a_n = a_{n-1} + \frac{1}{n^2}$$

is monotonically increasing.



There is a similar definition of a monotonically decreasing sequence.

Definition. A sequence is bounded above by a real number B if $a_n \leq B$ for all n.

Example. We will show that the sequence that we just defined recursively is bounded above by $B = 2$.

Theorem. A monotonically increasing sequence a_n that is bounded above by the number B converges, and the limit satisfies

$$\lim_{n\to\infty} a_n \leq B.$$

Remark. In the example above, it turns out that the limit is

$$\frac{\pi^2}{6} \approx 1.64,$$

but we will not verify this fact in this course.

Growth rates of sequences

Definition. Suppose that a_n and b_n are two sequences such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty.$$

Then b_n grows faster than a_n if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Notation: Our textbook uses the notation $a_n << b_n$ to indicate that b_n grows faster than a_n.

Fact: From last semester, we know that $\ln n << n^p << b^n << n^n$ where p is any positive power and b is any base that is greater than 1.

This semester we add an important new sequence to the list.

Example. The factorial sequence $n!$ can be defined recursively as

$$0! = 1 \quad \text{and} \quad n! = (n)(n-1)!$$

Note that $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, ...

A remarkable curiosity. The number of seconds in six weeks is exactly 10!

Consequently, 11! is greater than the number of seconds in a year.

12! is greater than the number of seconds in 12 years.

13! is greater than the number of seconds in a century.

Question: Where does the factorial sequence $n!$ fit among our growth rates of common sequences?

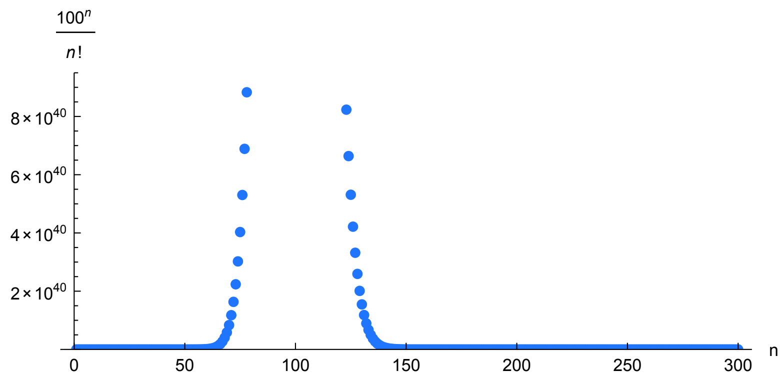
Let's compare b^n and $n!$ in the case where $b = 100$. For example,

$$\frac{100^{30}}{30!} = \frac{1907348632812500000000000000000000000000000000}{50592967951238834121} \approx 3.77 \times 10^{27}.$$

If we plot the ratio

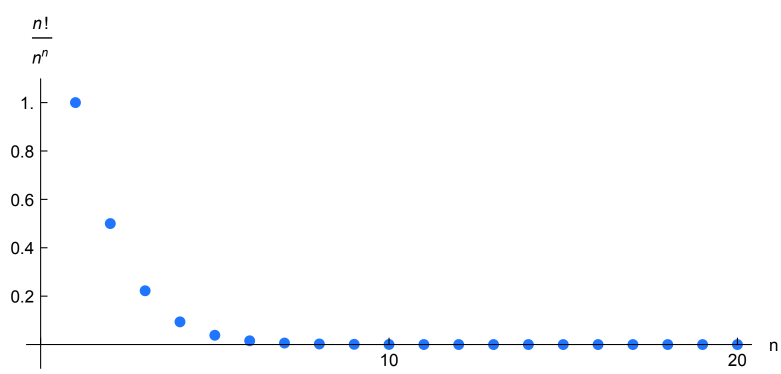
$$\frac{100^n}{n!}$$

for $1 \leq n \leq 300$, we get the following figure.



The figure suggests that $100^n << n!$. Here is how we can verify this fact:

Now let's compare $n!$ with n^n. We plot $\dfrac{n!}{n^n}$.



We conclude that $\ln n << n^p << b^n << n! << n^n$.