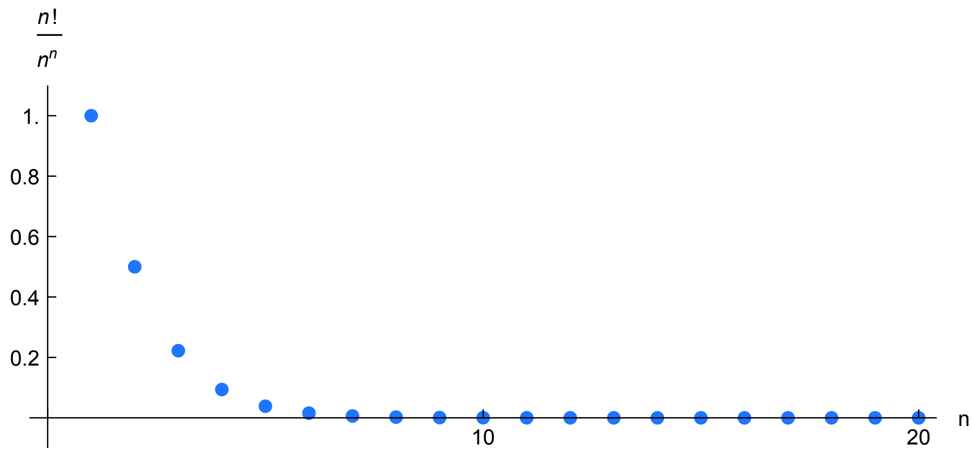
A little more on growth rates of sequences

Last class we saw that $100^n << n!$. You should think about the fact that there is nothing special about 100. The base 100 can be replaced by any base $b > 1$.

Now let's compare $n!$ with n^n. First, we plot $\frac{n!}{n^n}$.



We conclude that $\ln n << n^p << b^n << n! << n^n$.

Infinite series

We begin with an example. Let $x = 0.9999\ldots$.

To understand this computation, we need the concept of an infinite series. An **infinite series** is the sum of an infinite list of numbers. That is, an infinite series is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \dots .$$

How do we determine if such a sum makes sense?

We consider the sequence of **partial sums**. Given an infinite series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \dots ,$$

we define its sequence of partial sums $\{S_n\}$ by

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \end{aligned}$$

Notation: Be careful about the difference between the terms a_k of an infinite series and its nth partial sums S_n.

Example. Consider the infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots .$$

Remark. Note that the sequence of partial sums for any series can be defined recursively by

$$S_n = S_{n-1} + a_n \, .$$

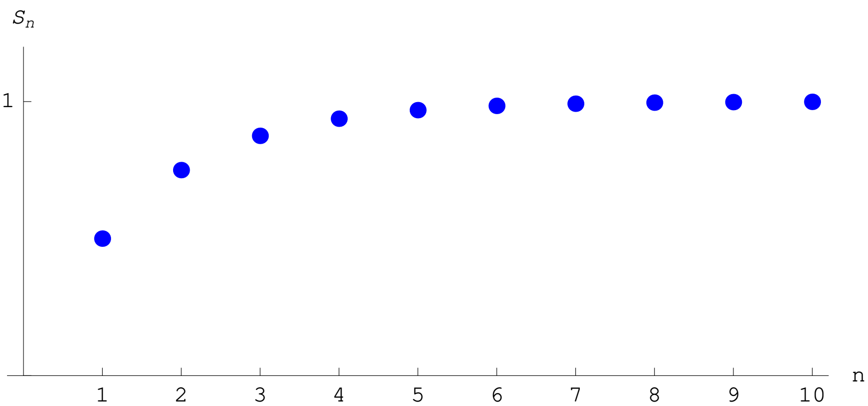
Definition. The infinite series $a_1 + a_2 + a_3 + \ldots$ **converges** if the limit

$$\lim_{n\to\infty} S_n$$

exists and is finite. Otherwise, the infinite series **diverges.**

Example. Here is a picture of the sequence of partial sums for the series

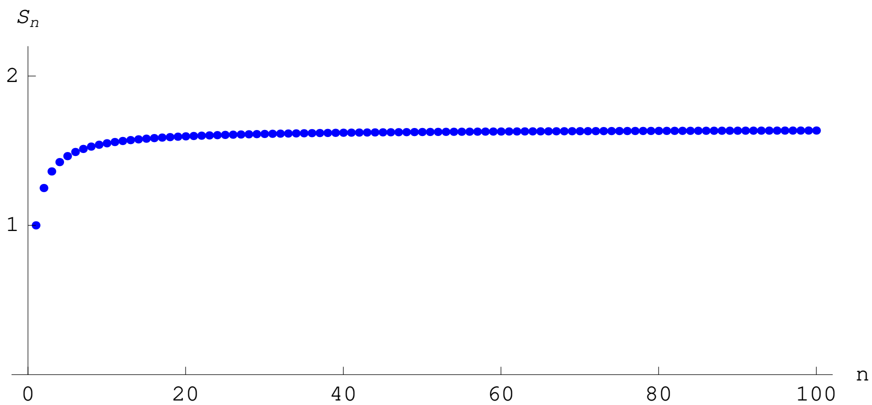
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \, .$$



Example. Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \, .$$

Last class we discussed the fact that its sequence of partial sums is monotonically increasing.



Example. Consider the infinite series $1 + 1 + 1 + 1 + \dots$.

Example. Consider the infinite series $1 - 1 + 1 - 1 \pm \dots$.

Geometric series

Definition. A geometric series is one in which the ratio of successive terms is constant. In other words, there is a number r such that

$$\frac{a_{n+1}}{a_n} = r$$

for all n.

Example. The series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ is a geometric series.

Example. The decimal expansion $x = 0.999\dots$ is also a geometric series.

Example. The series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ is not geometric.

Geometric series are nice because we can always find a formula for the sequence of partial sums. If we write the series as

$$a + ar + ar^2 + \dots,$$

then we have

Theorem. Consider the geometric series $a + ar + ar^2 + \dots$ where $a \neq 0$.

- If $|r| < 1$, then the series converges to $\dfrac{a}{1-r}$.
- If $|r| \geq 1$, then the series diverges.

Example. We return to the decimal expansion $x = 0.999\dots$.

Example. What about $x = 3.142857\overline{142857}$?