A little more on growth rates of sequences

Last class we saw that $100^n \ll n!$. You should think about the fact that there is nothing special about 100. The base 100 can be replaced by any base $b > 1$.

Now let’s compare $n!$ with $n^n$. First, we plot $\frac{n!}{n^n}$.

We conclude that $\ln n \ll n^p \ll b^n \ll n! \ll n^n$.

Infinite series

We begin with an example. Let $x = 0.9999 \ldots$
To understand this computation, we need the concept of an infinite series. An **infinite series** is the sum of an infinite list of numbers. That is, an infinite series is

\[
\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots
\]

How do we determine if such a sum makes sense? We consider the sequence of **partial sums**. Given an infinite series

\[
\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots
\]

we define its sequence of partial sums \( \{S_n\} \) by

\[
\begin{align*}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
\vdots
\end{align*}
\]

**Notation:** Be careful about the difference between the terms \( a_k \) of an infinite series and its \( n \)th partial sums \( S_n \).

**Example.** Consider the infinite series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots
\]
**Remark.** Note that the sequence of partial sums for any series can be defined recursively by

\[ S_n = S_{n-1} + a_n. \]

**Definition.** The infinite series \( a_1 + a_2 + a_3 + \ldots \) converges if the limit

\[ \lim_{n \to \infty} S_n \]

exists and is finite. Otherwise, the infinite series diverges.

**Example.** Here is a picture of the sequence of partial sums for the series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.
\]

**Example.** Consider the infinite series

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots.
\]

Last class we discussed the fact that its sequence of partial sums is monotonically increasing.
Example. Consider the infinite series $1 + 1 + 1 + 1 + \ldots$

Example. Consider the infinite series $1 - 1 + 1 - 1 \pm \ldots$

Geometric series

**Definition.** A geometric series is one in which the ratio of successive terms is constant. In other words, there is a number $r$ such that

$$\frac{a_{n+1}}{a_n} = r$$

for all $n$.

**Example.** The series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is a geometric series.

**Example.** The decimal expansion $x = 0.999\ldots$ is also a geometric series.

**Example.** The series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ is not geometric.
Geometric series are nice because we can always find a formula for the sequence of partial sums. If we write the series as

\[ a + ar + ar^2 + \ldots , \]

then we have

**Theorem.** Consider the geometric series \( a + ar + ar^2 + \ldots \) where \( a \neq 0 \).

- If \(|r| < 1\), then the series converges to \( \frac{a}{1-r} \).
- If \(|r| \geq 1\), then the series diverges.

**Example.** We return to the decimal expansion \( x = 0.999\ldots \).

**Example.** What about \( x = 3.142857142857\ldots \)?