Geometric series

Recall the definition of a geometric series from the end of last class

Definition. A geometric series is one in which the ratio of successive terms is constant. In other words, there is a number r such that

$$\frac{a_{k+1}}{a_k} = r \text{ for all } k.$$

Learning Catalytics exercise: Here's some space in case you need to do a quick calculation or want to take some notes after the exercise.

Geometric series are nice because we can always find a formula for the sequence of partial sums. If we write the series as

$$a + ar + ar^2 + \dots,$$

then we have

Theorem. Consider the geometric series $a + ar + ar^2 + \dots$ where $a \neq 0$.

- If $|r| < 1$, then the series converges to $\dfrac{a}{1-r}$.
- If $|r| \geq 1$, then the series diverges.

Example. We return to the decimal expansion $x = 0.999\ldots$.

Example. What about $x = 3.142857\overline{142857}$?

$$x = 3 + \frac{142857}{1000000} + \frac{142857}{1000000^2} + \frac{142857}{1000000^3} + \ldots = 3 + \frac{142857}{1000000}\left(\frac{1}{1 - \dfrac{1}{1000000}}\right)$$

$$= 3 + \frac{142857}{1000000}\left(\frac{1}{\dfrac{999999}{1000000}}\right) = 3 + \frac{142857}{1000000}\left(\frac{1000000}{999999}\right) = 3 + \frac{142857}{999999} = 3 + \frac{1}{7} = \frac{22}{7}.$$

More on the convergence of infinite series

The example above illustrates a basic principle regarding the convergence of a series: The convergence of a series does not depend on a finite number of terms added to or removed from the series. We say that the convergence of a series depends only on its "tail."

Theorem. Suppose that N is a positive integer. Then the two series

$$\sum_{k=1}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=N}^{\infty} a_k$$

either both converge or both diverge.

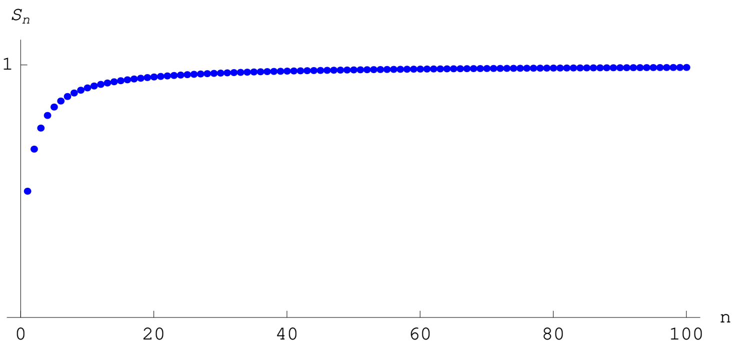
Remark. If an infinite series converges, its value depends on all of the terms of the series. However, the question of convergence depends only the tail of the series.

Geometric series are one of the few types of series for which we can calculate a closed form for the nth partial sum, but there are others.

Example. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots .$$

Does it converge? If so, what does it converge to?

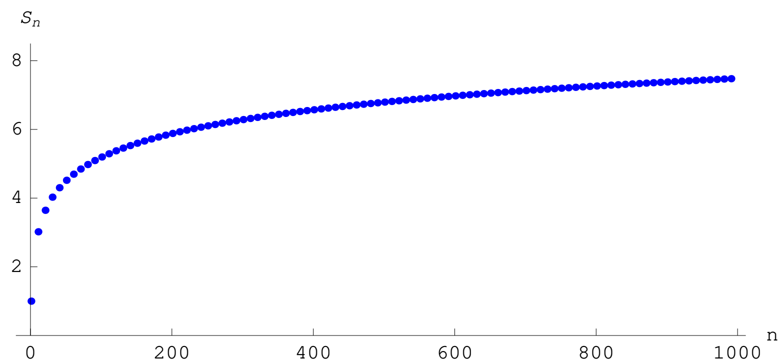


Sometimes it is very difficult to tell if a series converges by looking at a graph of its partial sums.

Example. The harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Here is a graph of its nth partial sums.



Does the harmonic series converge?