Taylor polynomials

Recall that our goal in the second half of this course is making sense of expressions such as

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

and

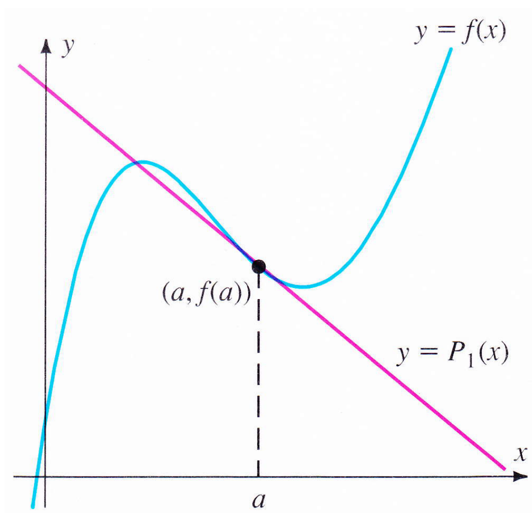
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots .$$

The approximation (as opposed to the infinite series) is one instance of Taylor approximation. We approximate a complicated function, e^x, by a cubic polynomial.

Linear approximation

Linear approximation uses the tangent line to the graph of a function to approximate the function. That is,

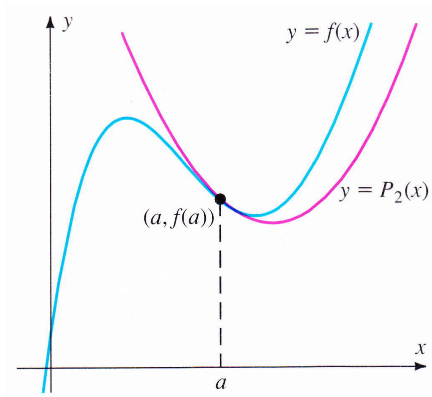
$$L(x) = f(a) + f'(a)(x - a).$$



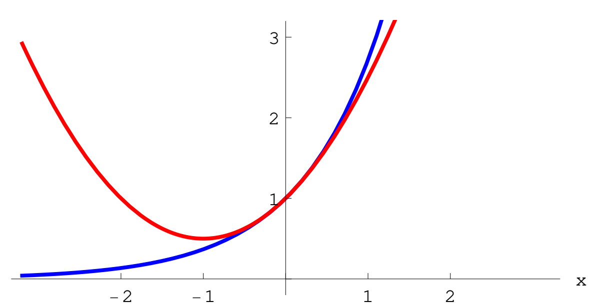
Taylor approximation is a generalization of this idea where we use polynomials of various degrees in addition to linear functions.

Quadratic approximation

When we approximate a function f at $x = a$ using a quadratic polynomial, we are the quadratic whose graph best "fits" the graph of f at $x = a$ (see the figure on the top of the next page).

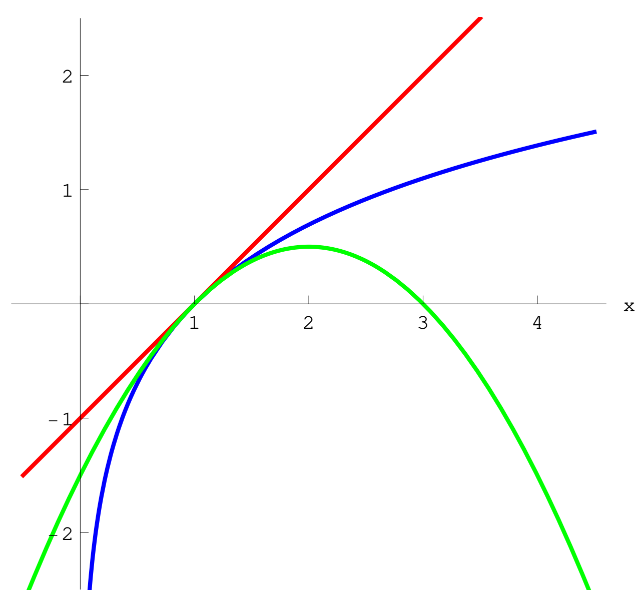


Example. Approximate the function $f(x) = e^x$ near $a = 0$ using a quadratic polynomial.



Sometimes approximating about $a = 0$ does not make sense.

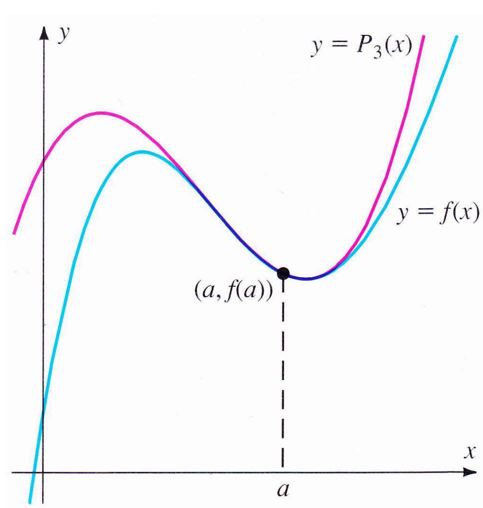
Example. Let $f(x) = \ln x$. Let's approximate f by linear and quadratic functions about $a = 1$.



General Taylor polynomials

Now consider the same type of approximation question with polynomials of higher degree and approximations near the number $x = a$. For example, let's approximate a three-times differentiable function f about $x = a$ with a polynomial of the form

$$p_3(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3.$$



Definition. Let f be a function that is n-times differentiable at $x = a$. Then the nth order Taylor polynomial of f centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example. Calculate Taylor polynomials for $f(x) = \sin x$ centered at $a = 0$.