Learning Catalytics exercise: Here's some space in case you need to do a quick calculation or want to take some notes when we finish the exercise.

More on Taylor polynomials

Exercise. Let a be a given real number and define

$$p_5(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5$$

where the c_k, $k = 0, 1, 2, 3, 4, 5$, stand for constants or coefficients (whichever word makes the most sense for you).

1. Express $p_5(a)$ in terms of the c_k.
2. Calculate $p_5'(x)$ and express $p_5'(a)$ in terms of the c_k.
3. Calculate $p_5''(x)$ and express $p_5''(a)$ in terms of the c_k.
4. Calculate $p_5'''(x)$ and express $p_5'''(a)$ in terms of the c_k.
5. Calculate $p_5^{(4)}(x)$ and express $p_5^{(4)}(a)$ in terms of the c_k.
6. Calculate $p_5^{(5)}(x)$ and express $p_5^{(5)}(a)$ in terms of the c_k.
7. Calculate $p_5^{(6)}(x)$.
8. How would this exercise change if you made the same computations with one more term added to the polynomial? In other words, suppose that you start with $p_6(x)$ rather than $p_5(x)$, where $p_6(x) = p_5(x) + c_6(x-a)^6$.

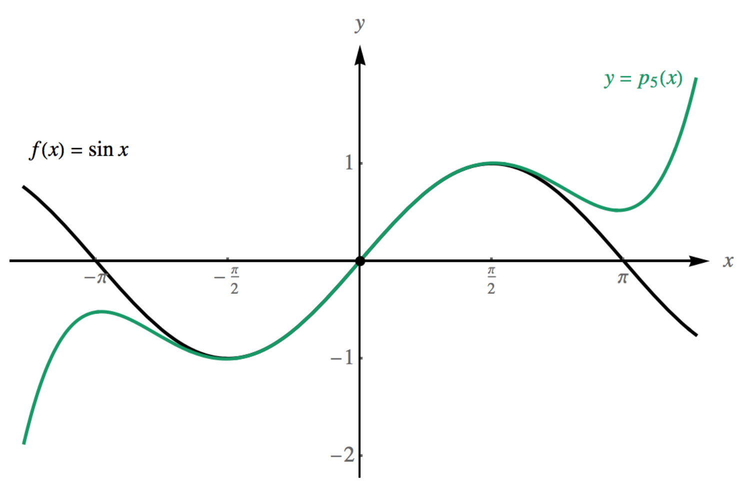
Definition. Let f be a function that is n-times differentiable at $x = a$. Then the nth order Taylor polynomial of f centered at a is

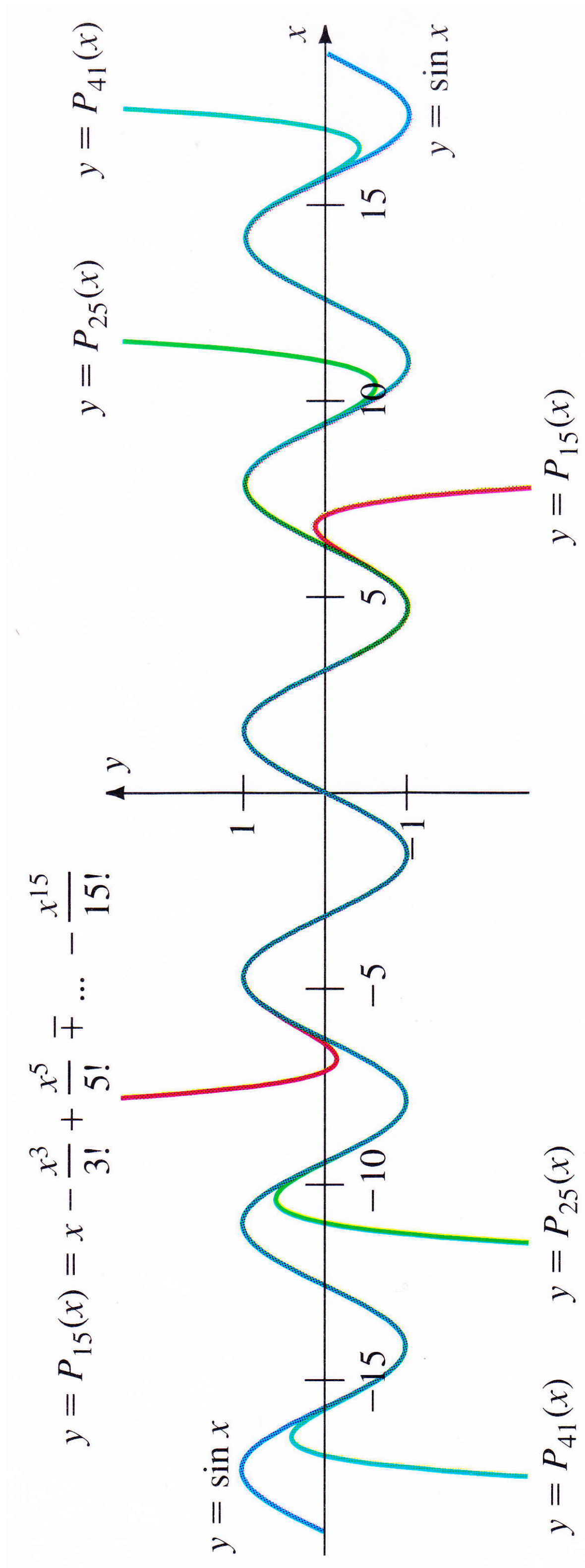
$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remark. Using the logic involved in the exercise, note that

$$p_n^{(k)}(a) = f^{(k)}(a) \text{ for } k = 0, 1, 2, \ldots, n.$$

Example. Calculate Taylor polynomials for $f(x) = \sin x$ centered at $a = 0$.

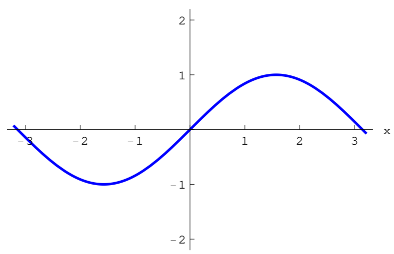
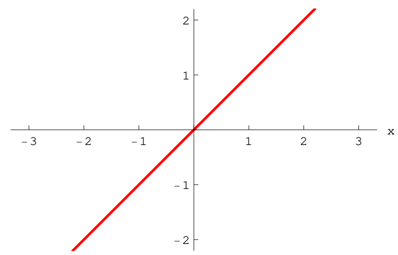
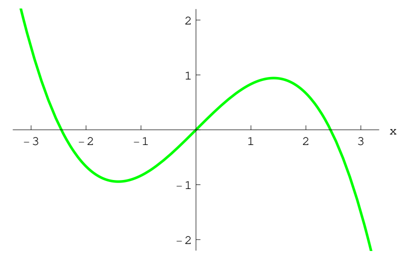


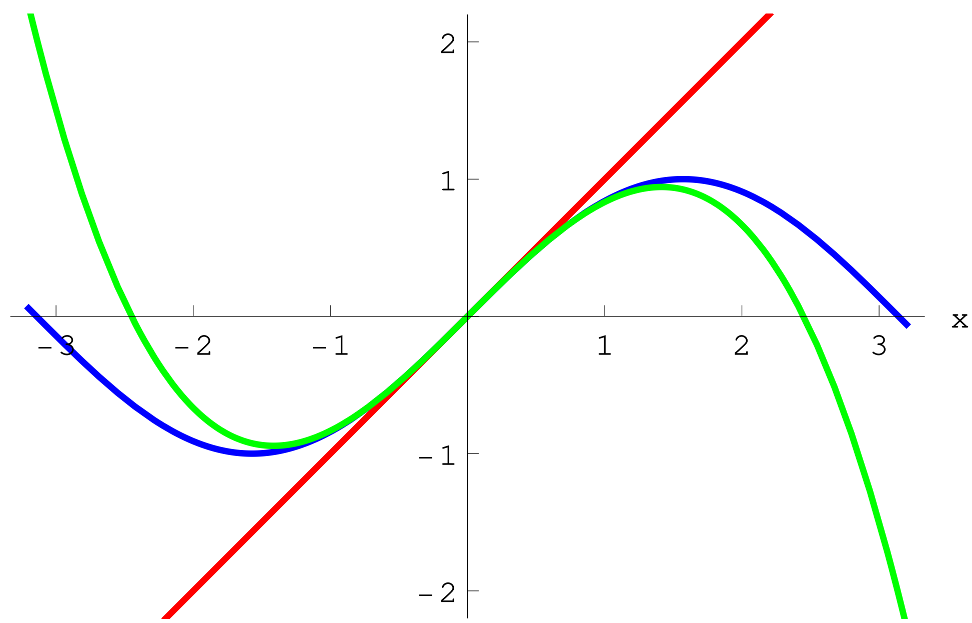

$y = P_{15}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots - \frac{x^{15}}{15!}$
$y = P_{25}(x)$
$y = P_{41}(x)$
$y = P_{15}(x)$
$y = \sin x$
x
y
1
−1
5
10
15
−5
−10
−15

Now we want to estimate how well a Taylor polynomial approximates its function.

Consider the function $f(x) = \sin x$ and the Taylor polynomials

$$p_1(x) = x \quad \text{and} \quad p_3(x) = x - \frac{x^3}{6}.$$



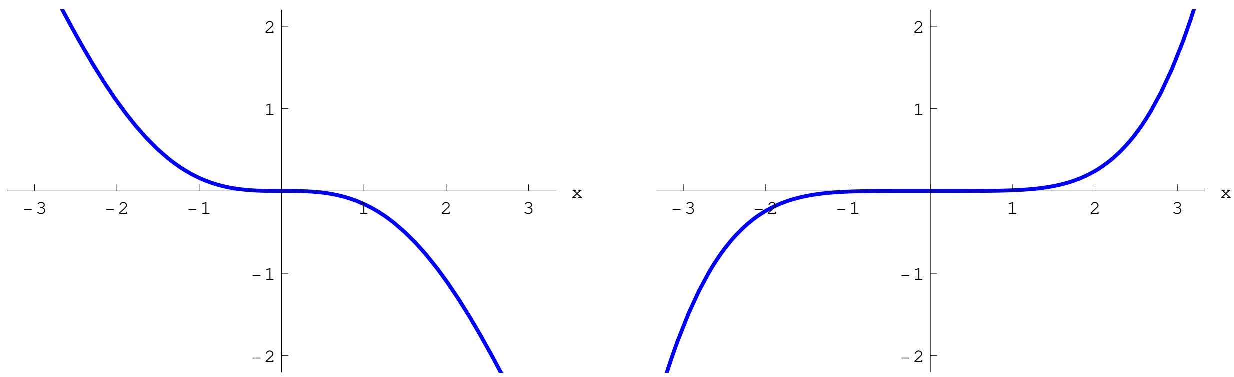
x	$x - \frac{x^3}{6}$	$\sin x$
0.00	0.00000000	0.00000000
0.01	0.00999983	0.00999983
0.02	0.01999870	0.01999870
0.03	0.02999550	0.02999550
0.04	0.03998930	0.03998930
.	.	.
.	.	.
.	.	.
0.25	0.24739600	0.24740400
0.26	0.25707100	0.25708100
0.27	0.26672000	0.26673100
.	.	.
.	.	.
.	.	.
0.50	0.47916700	0.47942600
0.51	0.48789200	0.48817700
0.52	0.49656500	0.49688000
.	.	.
.	.	.
.	.	.
1.00	0.83333300	0.84147100
1.01	0.83828300	0.84683200
1.02	0.84313200	0.85210800
.	.	.
.	.	.
.	.	.
1.50	0.93750000	0.99749866
1.51	0.93617483	0.99815247
1.52	0.93469867	0.99871044
.	.	.
.	.	.
.	.	.
2.00	0.66666667	0.90929743

It is easier to see the error in the approximation by graphing the remainder function.

Definition. Let p_n be the Taylor polynomial of order n for f. The remainder in using p_n to approximate f at the number x is

$$R_n(x) = f(x) - p_n(x).$$

For the sine function, the remainders R_1 and R_3 are graphed below:



In general, we can estimate the remainder using Taylor's Theorem.

Theorem. (Taylor's Theorem) Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c between x and a. Suppose there exists an upper bound M such that $|f^{(n+1)}(c)| \leq M$ for all c between a and x. Then

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$