Taylor Approximation

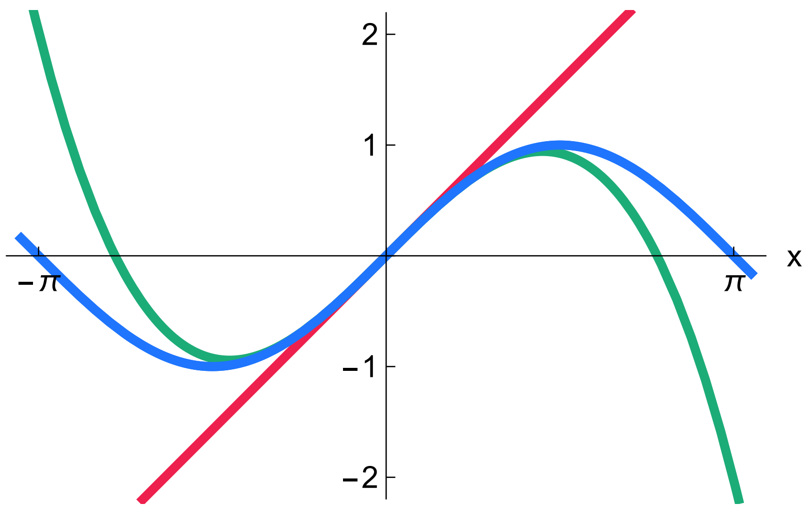
We want to estimate how well a Taylor polynomial approximates its function.

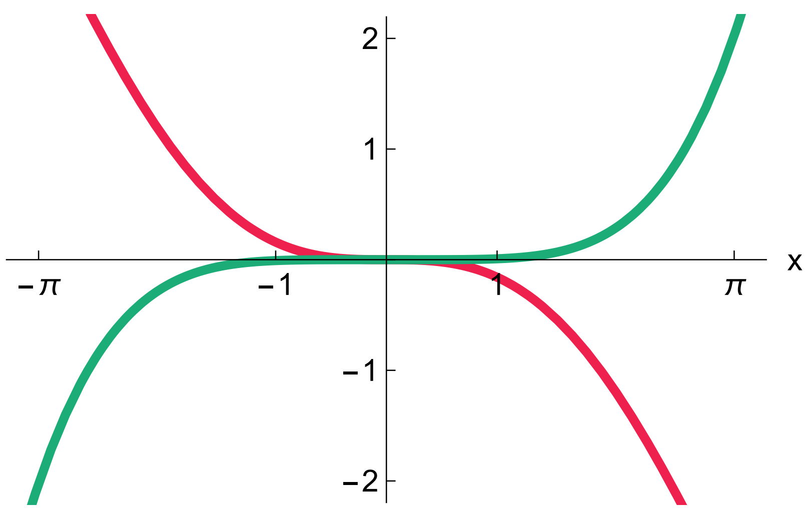
Definition. Let p_n be the Taylor polynomial of order n for f. The remainder in using p_n to approximate f at the number x is

$$R_n(x) = f(x) - p_n(x).$$

Example. Consider the function $f(x) = \sin x$ and the Taylor polynomials

$$p_1(x) = x \quad \text{and} \quad p_3(x) = x - \frac{x^3}{6}.$$





In general, we can estimate the remainder using Taylor's Theorem.

Theorem. (Taylor's Theorem) Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, the remainder is

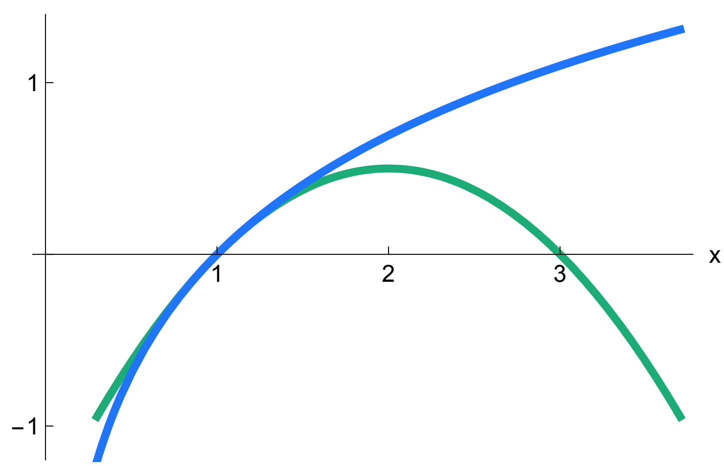
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c between x and a. Suppose there exists an upper bound M such that $|f^{(n+1)}(c)| \leq M$ for all c between a and x. Then

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example.

1. Approximate $\ln 1.1$ using the second-order Taylor polynomial of $\ln(x)$ centered at $a = 1$.
2. Use Taylor's Theorem to estimate the accuracy of that approximation.



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Example. How large must n be so that Taylor's Theorem guarantees that

$$|p_n(38) - \sqrt{38}\,| < 5 \times 10^{-5},$$

where $p_n(x)$ is the Taylor polynomial of order n centered at $a = 36$ for $\sqrt{x}\,$?

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