Power Series

Now that we understand what it means to say $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, we study what it means to write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

Example. Consider all geometric series whose first term is 1.

Definition. A power series centered at $a = 0$ is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots.$$

Note that its kth term is the product of a constant a_k and the expression x^k.

Example. The infinite series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

is the power series representation of the function $f(x) = \frac{1}{1-x}$ for $|x| < 1$.

First question: For what values of x does a power series represent a function?

Example. Consider the infinite series

$$\sum_{k=0}^{\infty} \frac{x^k}{2k+1} = 1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \dots .$$

Theorem. Given a power series

$$\sum_{k=0}^{\infty} a_k x^k,$$

there exists a number r with $0 \le r \le \infty$ such that

1. the power series converges absolutely if $|x| < r$ and
2. the power series diverges if $|x| > r$.

Definition. The number r is called the radius of convergence of the power series.

Note: The power series may or may not converge for $x = r$ or $x = -r$.

Example. Back to $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2k+1} = 1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \ldots.$

Example. Consider the power series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots .$$

Determine its radius and interval of convergence.

Power series can be centered at numbers a that are not zero. Such power series are written

$$\sum_{k=0}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots .$$

Example. Find the radius and interval of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k\,2^k}(x-2)^k.$$