Learning Catalytics exercise: Here's some space in case you need to do a quick calculation or want to take some notes when we finish the exercise.

More on power series

Last class we saw that a power series has a radius of convergence and an interval of convergence, and we were discussing the following example when class ended.

Example. Find the radius and interval of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k\,2^k}(x-2)^k.$$

We use the Ratio Test to show that this series converges absolutely if

$$|x-2| < 2.$$

Now we check the endpoints of the interval of convergence.

Producing new power series from known power series

We can produce new power series from known ones in predictable ways.

Theorem. Suppose that m is a positive integer, b is a real number, and $\sum a_k x^k$ is a power series that converges absolutely to the function $f(x)$ on the interval I. Then

1. The power series $x^m \sum a_k x^k = \sum a_k x^{k+m}$ converges absolutely to $x^m f(x)$ on I.
2. The power series $\sum a_k (bx^m)^k$ converges absolutely to $f(bx^m)$ on I if bx^m is in I.

Example. Find a power series representation for the function $g(x) = \dfrac{x}{1 - 3x^2}$.

Example. Consider the power series $\sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 \pm \dots$.

Example. Consider the power series $\sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + x^6 + \dots$. We have

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$

for $-1 < x < 1$.

Differentiating and integrating power series

We can differentiate power series term by term. Here is a more precise statement.

Theorem. Suppose that the power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

has a radius of convergence $r > 0$, and let f be the function to which it converges. Then

1. the function f is differentiable for $-r < x < r$,
2. the power series
$$\sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$
converges absolutely for each x in the interval $-r < x < r$, and
3. $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$ for $-r < x < r$.

Example. Apply this theorem to the geometric series

$$1 + x + x^2 + x^3 + \dots .$$

Example. Recall that the power series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

has an infinite radius of convergence.

We can also integrate power series term by term.

Theorem. Suppose that the power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

has a radius of convergence $r > 0$, and let f be the function to which it converges. Then

1. the power series
$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots$$
converges absolutely for each x in the interval $-r < x < r$, and

2. $\displaystyle\int f(x)\,dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots$ for $-r < x < r$.

Example. Apply this theorem to the power series expansion for $f(x) = \dfrac{1}{1+x}$.