Integrating power series

We can integrate power series term by term.

Theorem. Suppose that the power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

has a radius of convergence $r > 0$, and let f be the function to which it converges. Then

1. the power series
$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots$$
converges absolutely for each x in the interval $-r < x < r$, and

2. $\displaystyle\int f(x)\,dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots$ for $-r < x < r$.

Example. Apply this theorem to the power series expansion for $f(x) = \dfrac{1}{1+x}$.

Taylor series

Now that we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \text{ and } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \text{ if } |x| < 1.$$

Let's see how these series are related to the Taylor polynomials for these functions.

Theorem. If a function f is equal to the power series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

with a positive radius of convergence about $x = a$, then $c_k = \frac{f^{(k)}(a)}{k!}$ for all k.

Definition. Suppose that the function f is infinitely differentiable in some open interval centered at the number a. Then the Taylor series for f centered at a is the power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots .$$

A Taylor series that is centered at $a = 0$ is sometimes called a Maclaurin series.

Note. The Taylor series for a function is an "infinitely long" Taylor polynomial. In other words, to obtain the Taylor polynomial of order n for a given function, we truncate the Taylor series by removing all terms whose degrees are greater than n.

Examples. We have already done the calculations necessary to compute some very important Maclaurin series. On April 5, we basically showed that the Maclaurin series for $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \dots .$$

Last class we showed that the Maclaurin series for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Example. Compute the Maclaurin series for $\cos x$.

Example. Compute the Maclaurin series for $x^3e^{x^2}$.

Example. Compute the ninth-order Taylor polynomial centered at 0 for $x^3e^{x^2}$.

Example. Compute the Maclaurin series for $\sin^2 x$.

Example. Compute the Taylor series centered at 2 for $1/x$.

Logical slight of hand. You may be assuming that these functions are equal to their Taylor (Maclaurin) series on their intervals of convergence. In the cases discussed in this class, that is indeed true. Unfortunately, there are functions that are infinitely differentiable but **do not** equal their series. We will discuss this unfortunate fact in the near future.