More on Taylor series

Example. Compute the Taylor series centered at 2 for $1/x$.

Logical slight of hand. You may be assuming that these functions are equal to their Taylor (Maclaurin) series on their intervals of convergence. In the cases discussed in this class, that is indeed true. Unfortunately, there are functions that are infinitely differentiable but **do not** equal their series. We will discuss this unfortunate fact in the near future.

The binomial series

Newton (among others) realized that we can use power series to compute the values of $f(x) = (1+x)^p$ for any real value of p. For example, we can use Taylor series to compute roots such as square roots ($p = 1/2$) and cube roots ($p = 1/3$).

We begin by reviewing binomial expansion as you may have seen it in your prior algebra courses. In particular, suppose we want to expand the expression

$$(a+b)^n$$

where n is a positive integer. The binomial formula states that

$$(a+b)^n = a^n + c_1 a^{n-1} b + c_2 a^{n-2} b^2 + \ldots + c_{n-1} a b^{n-1} + b^n.$$

The coefficients c_k can be described using Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

These coefficients can also be generated using the "n choose k" notation

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

from the study of combinatorics. "n choose k" is the number of subsets of k objects from a set of n distinct objects.

Using this notation, we can rewrite the binomial formula as

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + b^n.$$

If we factor out a and let $x = b/a$, we have

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \ldots + \binom{n}{k} x^k + \ldots + x^n.$$

Newton realized that the binomial formula in this form could be generalized to arbitrary exponents by using Taylor series.

Definition. Given a real number p and a positive integer k,

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} \quad \text{and} \quad \binom{p}{0} = 1.$$

Example. Compute $\binom{\frac{1}{3}}{3}$.

We obtain the binomial series by calculating the Maclaurin series for the function

$$f(x) = (1+x)^p$$

where p is any real number.

Theorem. Given a real number p, the power series

$$\sum_{k=0}^{\infty} \binom{p}{k} x^k = \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k$$

converges to the function $f(x) = (1+x)^p$ for $|x| < 1$.

Example. Compute the binomial series for $f(x) = \dfrac{1}{\sqrt{1+x}}$.

Example. Use the first four terms of a binomial series to approximate $\sqrt[5]{1.1}$