Learning Catalytics exercise: Here's some space in case you need to do a quick calculation or want to take some notes when we finish the exercise.

Working with Taylor series

Sometimes power series are very handy for the computation of limits.

Example. Calculate $\lim_{x \to 0} \frac{3\tan^{-1} x - 3x + x^3}{x^5}$.

We can use power series to approximate definite integrals.

Example. Estimate the integral $\int_0^1 e^{-x^2}\,dx$ with an error no larger than 0.01.

Convergence of Taylor series

Logical slight of hand. You may be assuming that these functions are equal to their Taylor (Maclaurin) series on their intervals of convergence. In the cases discussed in this class, that is indeed true. Unfortunately, there are functions that are infinitely differentiable but **do not** equal their series. We will discuss this unfortunate fact now.

Question: Suppose that the Taylor series (or Maclaurin series) of a function f centered at $x = a$ has a positive radius of convergence. Does the series converge to f?

The answer to this question relies on Taylor's Theorem. Recall that the nth remainder function is

$$R_n(x) = f(x) - p_n(x)$$

where p_n is the nth degree Taylor polynomial of f centered at $x = a$.

Theorem. (Taylor's Theorem) Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

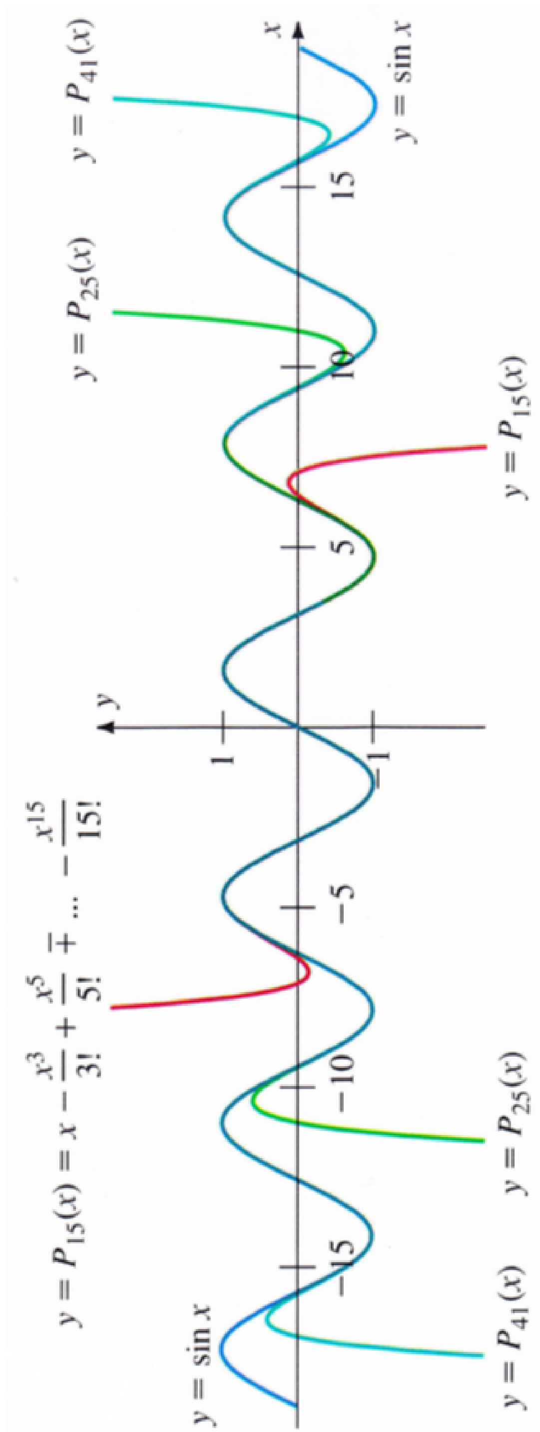
for some point c between x and a. Suppose there exists an upper bound M such that $|f^{(n+1)}(c)| \leq M$ for all c between a and x. Then

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Theorem. Let f have derivatives of all orders on an open interval I containing the number a. The Taylor series for f centered at $x = a$ converges to f for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.


$y = P_{41}(x)$
$y = P_{25}(x)$
$y = P_{15}(x)$
$y = \sin x$
$y = P_{15}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots - \frac{x^{15}}{15!}$
$y = \sin x$
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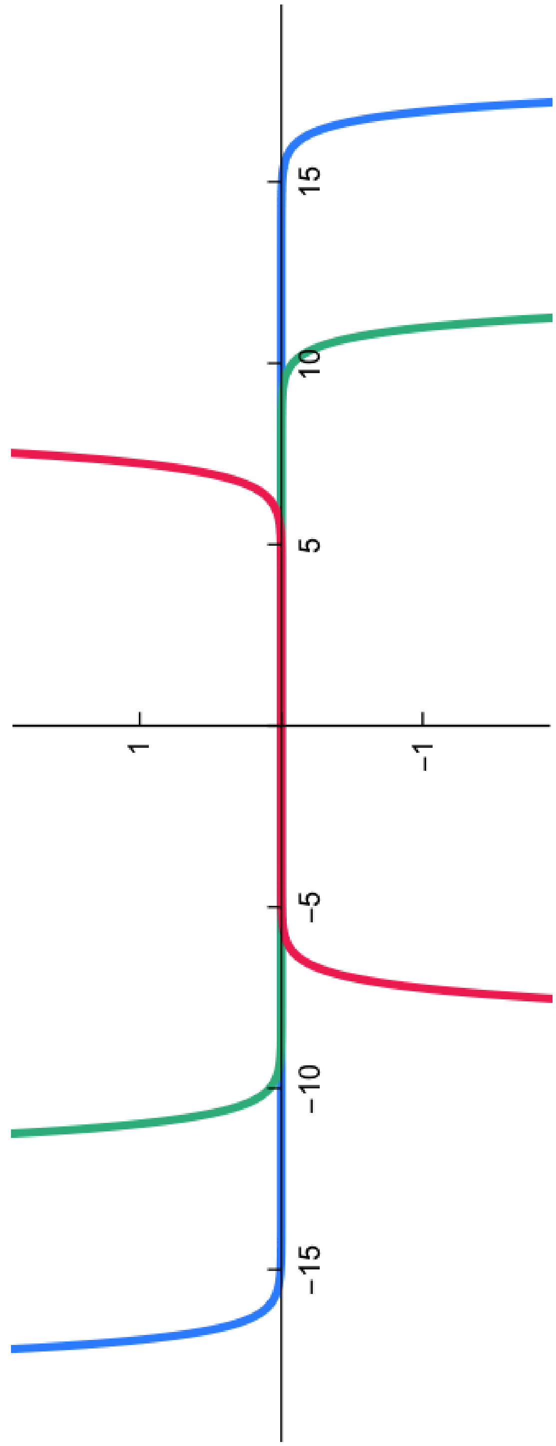
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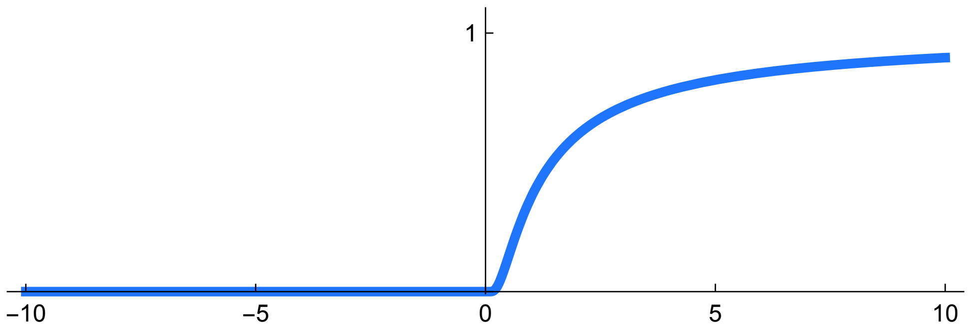
Example. Show that $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for all real numbers x.

It is not always true that the Taylor series of a function f converges to f.

Example. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{(-1/x)} & \text{if } x > 0. \end{cases}$$



It can be shown that $f^{(k)}(0) = 0$ for $k = 0, 1, 2, 3, \ldots$ (definitely a challenge exercise). Therefore, the Taylor series of f is just the function that is constantly zero.