A little review of the Method of Substitution

The most important technique of integration is the Method of Substitution, and this class will be devoted to a review of this method. However, I want to make one comment about notation first.

What is the difference between $\int_a^b f(x) \, dx$ and $\int f(x) \, dx$?

**Example.**

$$\int_0^{3\pi/2} \sin x \, dx = \quad \text{and} \quad \int \sin x \, dx =$$

The Method of Substitution

The basic idea behind the method is to make a change of variables that converts one integral into another, more “basic” one. The method is based on the Chain Rule.

**Example.** Consider the function $\sin(x^2)$. Then

$$\frac{d}{dx} \sin(x^2) =$$

We obtain the integral

Here’s how this kind of calculation is done in more abstract notation.

$$\frac{d}{dx} F(g(x)) =$$

We obtain the integral

In the example, $F(x) = \sin(x)$ and $g(x) = x^2$.

We often refer to $g(x)$ as a change of variables, and we write $u = g(x)$. Then

$$\frac{d}{dx} F(u) = F'(u) \frac{du}{dx}.$$ 

So

$$\int F'(u) \left( \frac{du}{dx} \right) \, dx = F(u) + C.$$
Differential notation: We define the differential

\[ du = \frac{du}{dx} \, dx. \]

Then we get

\[ \int F'(u) \, du = F(u) + C. \]

**Example.** \[ \int x \sqrt{9 - x^2} \, dx \]

You can always check your answer!
Example. $\int_0^3 x\sqrt{9-x^2} \, dx$

We will do this definite integral two (different) ways.

1. Calculate the indefinite integral $\int x\sqrt{9-x^2} \, dx$. Then evaluate.

2. Change the limits of integration when you substitute.
The geometric interpretation of this computation is that the areas under the following two graphs are equal. The graph on the left is the graph of \( f(x) = x\sqrt{9 - x^2} \) and the graph on the right is the graph of \( F(u) = \frac{1}{2}\sqrt{u} \).

**Example.** \( \int (\tan x)(\sec^2 x) \, dx \)