Infinite series—a brief review

An infinite series is the sum of an infinite list of numbers. That is, an infinite series is

\[
\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots .
\]

We consider the sequence of partial sums. That is, we define its sequence of partial sums \( \{S_n\} \) by

\[
\begin{align*}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
& \vdots
\end{align*}
\]

**Definition.** The infinite series \( a_1 + a_2 + a_3 + \ldots \) converges if the limit

\[
\lim_{n \to \infty} S_n
\]

exists and is finite. Otherwise, the infinite series diverges.

Geometric series

**Definition.** A geometric series is one in which the ratio of successive terms is constant. In other words, there is a number \( r \) such that

\[
\frac{a_{n+1}}{a_n} = r
\]

for all \( n \).

**Theorem.** Consider the geometric series \( a + ar + ar^2 + \ldots \) where \( a \neq 0 \).

- If \( |r| < 1 \), then the series converges to \( \frac{a}{1 - r} \).
- If \( |r| \geq 1 \), then the series diverges.

We can also determine the sum of a series if it is a telescoping series.
Tests for convergence

Most infinite series do not yield explicit expressions for their $n$th partial sums $S_n$. Therefore, we concentrate on “tests for convergence” that do not require that we determine a formula for $S_n$.

The most basic of these tests is the $k$th term test. It is also known as the divergence test.

**Theorem.** If the infinite series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

**The $k$th Term Test for Divergence:** If $\lim_{k \to \infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

**Remark.** It is important to remember the harmonic series when one thinks about how the $k$th term test is used. The $k$th term test never establishes convergence of a series. It can only be used to conclude that a series diverges.

Convergence tests for positive term series

Suppose that we have a series $a_1 + a_2 + a_3 + \ldots$ where all of the terms $a_k$ are positive. Then the sequence of partial sums $S_n$ is a monotonically increasing sequence. That is,

$$S_1 < S_2 < S_3 < \ldots .$$

**Theorem.** A positive term series converges if and only if its sequence of partial sums is bounded above.

This theorem gives us a strategy for determining the convergence of a series with positive terms. We look for ways to bound the sequence of partial sums.
The Integral Test

**Theorem.** (Integral Test) Let \( a_k = f(k) \) for \( k = 1, 2, 3, \ldots \) where \( f \) is a function that is continuous, positive, and decreasing on the interval \([1,\infty)\). Then

\[
\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x) \, dx
\]

either both converge or both diverge. If they converge, the value of the integral is not, in general, the value of the series.

**p-series**

There is an important class of series, the \( p \)-series, whose convergence can be determined by the Integral Test. However, before we consider these series, we need to recall the family of improper integrals

\[
\int_1^{\infty} \frac{1}{x^p} \, dx
\]

where \( p \) is any real constant.

We calculated that

\[
\int_1^{\infty} \frac{1}{x^p} \, dx = \begin{cases} 
\text{converges to} & \frac{1}{p - 1} \quad \text{if} \quad p > 1 \\
\text{diverges} & \text{if} \quad p \leq 1.
\end{cases}
\]

Therefore, we can use the Integral Test to determine the convergence of all \( p \)-series.

**Theorem.** Consider the \( p \)-series \( \sum_{k=1}^{\infty} \frac{1}{k^p} \).

- It converges if \( p > 1 \), and
- it diverges if \( p \leq 1 \).
Examples. Consider the two series

1. \[ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots \]

2. \[ 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots \]

Which of these two series converge?

Approximating infinite series using the Integral Test

Suppose that the Integral Test applies to

\[ \sum_{k=1}^{\infty} a_k \]

where \( a_k = f(k) \) for some decreasing function \( f \). Recall that the \( n \)th partial sum is

\[ S_n = a_1 + a_2 + \ldots + a_n. \]

We can use improper integrals to approximate the infinite sum.

**Theorem.**

\[ S_n + \int_{n+1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_{n}^{\infty} f(x) \, dx \]

To see why, we sketch two figures:
Example. Estimate the accuracy of the approximation \( \sum_{k=1}^{\infty} \frac{1}{k^2} \approx \sum_{k=1}^{100} \frac{1}{k^2} \).

Example. How many terms of the series \( \sum_{k=1}^{\infty} \frac{1}{k^3} \) must be summed to estimate the value of the series within an error of \( 10^{-3} \)?