Alternating series

An alternating series is one in which the terms alternate between positive and negative numbers.

**Example.** \[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots\]

Here is a plot of its partial sums:
**Theorem 1.** Consider the alternating series

\[ \sum_{k=1}^{\infty} (-1)^{k+1}a_k \]

with \( a_k > 0 \). It converges if the following two conditions hold:

1. The \( a_k \) satisfy the condition that \( a_{k+1} \leq a_k \) for all \( k \).

2. \( \lim_{k \to \infty} a_k = 0 \).

If so, the series converges to a value \( S \) between 0 and \( a_1 \).

**Example.** \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 + 4} = \frac{1}{5} - \frac{1}{4} + \frac{3}{13} - \frac{1}{5} \pm \ldots \]

Here is a plot of its partial sums:

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Estimating the remainder in an alternating series
The $n$th remainder $R_n$ for a series that converges to $S$ is

$$R_n = |S - S_n|.$$  

**Theorem 2.** For an alternating series that satisfies the hypothesis of Theorem 1, then

$$R_n < a_{n+1}.$$
Example. How large must \( n \) be so that \( S_n \) approximates

\[
S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \ldots
\]

to at least one decimal place?

Example. Approximate the sum of the alternating series

\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k)!}
\]

to three decimal places.
Example. How many terms of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

do we need to sum to be sure that the remainder is less than $10^{-4}$? (We shall see that this infinite sum converges to $1/e$.)