Absolute convergence

Consider the three infinite series

\[
\begin{align*}
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} & \pm \ldots \\
1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} & \pm \ldots \\
1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} - \frac{1}{36} & + + - \ldots
\end{align*}
\]

Here are the partial sums for the third example:

\[
\sum_{n=1}^{10} s_n
\]

**Definition.** Let \( \sum a_k \) be an infinite series with positive and negative terms. We say that the series converges absolutely if the series

\[
\sum |a_k|
\]

converges. If \( \sum a_k \) converges but does not converge absolutely, then we say that it converges conditionally.

**Examples.** Consider the three infinite series above.

\[
\begin{align*}
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} & \pm \ldots \\
1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} & \pm \ldots \\
1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} - \frac{1}{36} & + + - \ldots
\end{align*}
\]
**Theorem.** If the series $\sum a_k$ converges absolutely, then it converges.

**Example.** Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{\cos 1.2k}{k^2} \approx .36 - \frac{.74}{4} - \frac{.90}{9} + \frac{.09}{16} + \frac{.96}{25} + \frac{.61}{36} - \frac{.52}{49} - \frac{.99}{64} \cdots.$$ 

Here are the partial sums for this example:
We can modify the Ratio Test so that it tests for absolute convergence.

**Theorem.** For a series \( \sum a_k \) with positive and negative terms, let

\[
r = \lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|}.
\]

If \( r < 1 \), then the series converges absolutely.

**Example.** Consider the alternating series

\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k + 1)^2}{(2k)!} = 2 - \frac{3}{8} + \frac{1}{45} - \frac{5}{8064} + \ldots
\]

Here are the partial sums for this example:

![Graph showing the partial sums of the series](image)

**Crazy Fact:** If a series converges absolutely, then its terms may be summed in any order without changing the value of the series. However, if the series converges conditionally, then the value of the series depends on the order of the terms.