Convergence of infinite series

Last class we proved that geometric series with ratios \( r \) such that \( |r| < 1 \) converge.

**Theorem.** Consider the geometric series \( a + ar + ar^2 + \ldots \) where \( a \neq 0 \).

- If \( |r| < 1 \), then the series converges to \( \frac{a}{1 - r} \).
- If \( |r| \geq 1 \), then the series diverges.

**Example.** We return to the decimal expansion \( x = 0.999 \ldots \).

**Example.** Another repeating decimal expansion \( x = 0.666 \ldots \).

**Example.** What about \( x = 3.142828 \ldots ? \)

This last example illustrates a basic principle regarding the convergence of series: The convergence of a series does not depend on a finite number of terms added to or removed from the series. We say that the convergence of a series depends only on its “tail.”

**Theorem.** Suppose that \( N \) is a positive integer. Then the two series

\[
\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=N}^{\infty} a_k
\]

either both converge or both diverge.
Geometric series are one of the few types of series for which we can calculate a closed form for the $n$th partial sum, but there are others.

**Example.** Consider the series

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots .$$

Does it converge? If so, what does it converge to?
Example. \[ \sum_{k=1}^{\infty} \ln \left( \frac{k}{k+1} \right) \]

Sometimes it is very difficult to tell if a series converges by looking at a graph of its partial sums.

Example. The harmonic series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]

Here is a graph of its nth partial sums.
Does the harmonic series converge?

Tests for convergence
Most infinite series do not yield explicit expressions for their \( n \)th partial sums \( S_n \). Therefore, we concentrate on “tests for convergence” that do not require that we determine a formula for \( S_n \).

The most basic of these tests is the \( k \)th term test. Suppose that the infinite series

\[
\sum_{k=1}^{\infty} a_k
\]

converges.

**Theorem.** If the infinite series \( \sum_{k=1}^{\infty} a_k \) converges, then \( a_k \to 0 \) as \( k \to \infty \).
The $k$th Term Test for Divergence: If $\lim_{k \to \infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Example. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$

Remark. It is important to remember the harmonic series when one thinks about how the $k$th term test is used. The $k$th term test never establishes convergence of a series. It can only be used to conclude that a series diverges.