Tests for convergence

Unlike geometric series and telescoping series, most infinite series do not yield explicit expressions for their $n$th partial sums $S_n$. Therefore, we concentrate on "tests for convergence" that do not require that we determine a formula for $S_n$.

The most basic of these tests is the $k$th term test. Suppose that the infinite series

$$\sum_{k=1}^{\infty} a_k$$

converges.

**Theorem.** If the infinite series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

This theorem (and a little logic) yield the $k$th Term Test for Divergence.

**The $k$th Term Test for Divergence:** If $\lim_{k \to \infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

**Example.** $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$
Remark. It is important to remember the harmonic series when one thinks about how the kth term test is used. The kth term test never establishes convergence of a series. It can only be used to conclude that a series diverges.

Convergence tests for positive term series

Suppose that we have a series

\[ a_1 + a_2 + a_3 + \ldots \]

where all of the terms \( a_k \) are positive. Then the sequence of partial sums \( S_n \) is a monotonically increasing sequence. That is,

\[ S_1 < S_2 < S_3 < \ldots \]

**Theorem.** A positive term series converges if and only if its sequence of partial sums is bounded above.

**Example.** Note that this theorem does not hold for arbitrary series. For example, the partial sums of

\[ 1 - 1 + 1 - 1 + 1 + \ldots \]

are bounded above by \( M = 1 \). Nevertheless, this series diverges.

This theorem gives us a strategy for determining the convergence of a series with positive terms. We look for ways to bound the sequence of partial sums.
The Integral Test

One way to bound the sequence of partial sums is to use improper integrals. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots.$$ 

Now consider the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$
Theorem. (Integral Test) Let \( a_k = f(k) \) for \( k = 1, 2, 3, \ldots \) where \( f \) is a function that is continuous, positive, and decreasing on the interval \([1, \infty)\). Then

\[
\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x) \, dx
\]

either both converge or both diverge. If they converge, the value of the integral is not, in general, the value of the series.

Example. \( \sum_{k=1}^{\infty} \frac{k}{k^2 + 1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \ldots \)