More on Taylor approximation

Last class we discussed Taylor’s Theorem for estimating the remainder when a Taylor polynomial is used to approximate a function.

**Theorem.** (Taylor’s Theorem) Let \( f \) have continuous derivatives up to \( f^{(n+1)} \) on an open interval \( I \) containing \( a \). For all \( x \) in \( I \), the remainder is

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},
\]

for some point \( c \) between \( x \) and \( a \). Suppose there exists an upper bound \( M \) such that \( |f^{(n+1)}(c)| \leq M \) for all \( c \) between \( a \) and \( x \). Then

\[
|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.
\]

**Example.**

1. Calculate the \( n \)th degree Taylor polynomial \( p_n \) of \( e^x \) centered at \( a = 0 \).

2. Use Taylor’s Theorem to determine a value of \( n \) such that \( p_n \) provides an approximation of \( e \) to three decimal places. (An approximation to “three decimal places” is one that has an error that is less than 0.0005.)

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Power Series

Now that we understand what it means to say
\[ e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \]
we study what it means to write
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots. \]

**Example.** Consider all geometric series whose first term is 1.
**Definition.** A power series centered at \( a = 0 \) is an infinite series of the form

\[
\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.
\]

Note that its \( k \)th term is the product of a constant \( a_k \) and the expression \( x^k \).

**Example.** The infinite series

\[
\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots
\]

is the power series representation of the function \( f(x) = \frac{1}{1-x} \) for \( |x| < 1 \).

First question: For what values of \( x \) does a power series represent a function?

**Example.** Consider the infinite series

\[
\sum_{k=0}^{\infty} \frac{x^k}{2k+1} = 1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \ldots.
\]

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**Theorem.** Given a power series

$$
\sum_{k=0}^{\infty} a_k x^k,
$$

there exists a number $r$ with $0 \leq r \leq \infty$ such that

1. the power series is absolutely convergent if $|x| < r$ and
2. the power series diverges if $|x| > r$.

**Definition.** The number $r$ is called the radius of convergence of the power series.

**Note:** The power series may or may not converge for $x = r$ or $x = -r$. 
Example. Calculate the interval of convergence of the power series

\[ \sum_{k=0}^{\infty} \frac{k}{3^k} x^k. \]