More on surface area

Example. Let's find the area of the portion of the saddle $z = y^2 - x^2$ that projects onto the unit circle $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

The integral formula for surface area can be expressed in differential notation. If we use the variable S to represent surface area, then the differential formulation of our integral formula is

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

if the surface is the graph of $z = f(x, y)$. We should not forget that this is equivalent to the formula

$$dS = |\mathbf{N}|\, dA$$

where the vector

$$\mathbf{N} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface.

Your textbook also has a more general definition of surface area for parametric surfaces (Definition 4 on p. 879). You should convince yourself that the definition that we discussed today is consistent with that formula.

Triple integrals

Integrating functions of three variables is very similar to integrating functions of two variables. Intuitively,

$$\iiint\limits_R f(x, y, z)\, dV = \begin{cases} 1. & \text{“sum” of } f(x, y, z) \text{ over } R \\ 2. & (\text{average value of } f(x, y, z) \text{ over } R)(\text{volume}(R)) \\ 3. & \text{four-dimensional “volume” under the graph of } f(x, y, z) \end{cases}$$

More precisely, the triple integral is a limit of Riemann sums. We partition the region R in space into small rectangular parallelopipids

For each little “cube” R_{ijk}, we choose a point P_{ijk} in it and then we calculate the sum

$$S_{lmn} = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P_{ijk})(\Delta x)(\Delta y)(\Delta z).$$

As l, m, and $n \to \infty$, S_{lmn} approaches a limit

$$\iiint_R f(x,y,z)\,dV$$

which we call the triple integral of the function $f(x,y,z)$ over the region R.

We calculate this limit by repeated integration just as we did to calculate double integrals.

Intuitively, triple integrals are no more complicated than double integrals, yet they are harder to compute because they are harder to set up. The difficulty comes when we try to set up these integrals over more general regions than rectangular boxes.

Example. Evaluate

$$\iiint_Q z\,dV$$

where Q is the region bounded by the cylinder $x^2 + z^2 = 9$, the plane $y + z = 3$, and the plane $y = 0$.